

# On two types of rank reversal

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- ① Introduction
- ② Literature on the rank reversal problem
- ③ “Definition” of the rank reversal problem
- ④ Some experiments . . .
- ⑤ Conclusions

- Some MCDM's suffer from the rank reversal problem

# Introduction

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  - Clear definition?
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  - A rank reversal will be induced for any method when changing weights

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- Some MCDM's suffer from the rank reversal problem
- $\exists$  some blur around the problem
  - Clear definition?
  - Lack of attention (to tackle the problem, to get knowlegde).
- Some DM's do not accept a method when no stability of the result can be guaranteed (especially when ranking options)

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- Still obscure subject: no mentioning of the subject in 1000 pages by Figueira et al. (2005)

*“There are two kinds of rank reversal discussed in the literature. The first has to do with lottery comparisons . . . The other kind . . . is rank reversal that may be due to the addition or deletion of alternatives. There are now myriads of real-life examples which show that there are cases where rank does and should be allowed to reverse and others which show that rank should not be allowed to reverse. . . .”*

T. Saaty (1997)

*When a non-optimal alternative in an alternative set is replaced by a worse one, the pairwise outranking relations related to it may be changed accordingly and the overall ranking of the whole alternative set, which depends on those pairwise outranking relations, may also be changed. The first change is reasonable when considering the fact that a non-optimal alternative has been replaced by a worse one. However, the second change is unreasonable and may cause undesirable rank reversals . . . ”*

Wang & Triantaphyllou (2008)



## Start:

- set of alternatives:  $\mathcal{A} = \{a_1, \dots, a_m\}$
- set of criteria  $\mathcal{F} = \{f_1, \dots, f_n\}$
- MCDM
- DM specific information (weights, parameters, ...)

⇒ **Result:** Ranking  $R$  over  $\mathcal{A}$  with or without ties, partial or complete

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**Rank reversal problem:** the relative position in  $R'$  of some  $a_i \in \mathcal{A}$  change due to the presence of  $a_{m+1}$

# Example

	Min	Min	Min	Max
	$f_1$	$f_2$	$f_3$	$f_4$
$a_1$	1	7.8	4	0.30
$a_2$	3	0.0	3	0.37
$a_3$	1	3.9	1	0.33
$a_4$	2	5.5	1	0.35
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SOMETHEE	$f_1$	$f_2$	$f_3$	$f_4$
Type crit.	I	III	I	V
q		0		0.04
p		1.8		0.08

# Result: SOMETHEE

$$\chi^+(a) : \chi^+(a_6) > \chi^+(a_2) > \chi^+(a_4) > \chi^+(a_1) > \chi^+(a_5) > \chi^+(a_3)$$

$$\chi^-(a) : \chi^-(a_1) < \chi^-(a_5) < \chi^-(a_6) < \chi^-(a_2) < \chi^-(a_4) < \chi^-(a_3)$$

$$\chi(a) : \chi(a_1) > \chi(a_6) > \chi(a_2) > \chi(a_5) > \chi(a_4) > \chi(a_3)$$

$$\text{with } \chi(a) = \chi^+(a) - \chi^-(a)$$

$\Rightarrow$  SOMETHEE I:  $\{a_1, a_6\} \succ \{a_2, a_5\} \succ a_4 \succ a_3$

$\Rightarrow$  SOMETHEE II:  $a_1 \succ a_6 \succ a_2 \succ a_5 \succ a_4 \succ a_3$

# Experiment 1

Consider the supplementary alternative:

$\text{EXP}_1$	$f_1$	$f_2$	$f_3$	$f_4$
$a_7$	2	2.5	3	0.35

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Results:

SOMETHEE I	$\{a_1, a_6\} \succ \{a_2, a_5\} \succ a_4 \succ a_3$
EXPERIMENT 1	$\{a_1, a_6\} \succ \{a_2, a_5\} \succ a_4 \succ a_7 \succ a_3$

SOMETHEE II	$a_1 \succ a_6 \succ a_2 \succ a_5 \succ a_4 \succ a_3$
EXPERIMENT 1	$a_6 \succ a_1 \succ a_2 \succ a_5 \succ a_4 \succ a_7 \succ a_3$

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Consider the supplementary alternative:

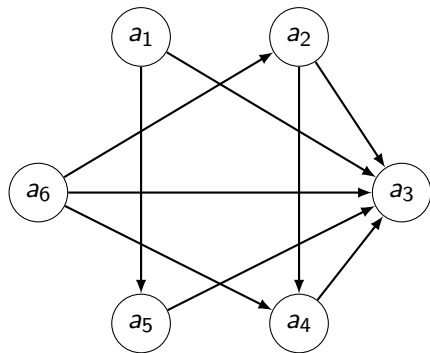
EXP <sub>1</sub>	f <sub>1</sub>	f <sub>2</sub>	f <sub>3</sub>	f <sub>4</sub>
a <sub>7</sub>	2	2.5	3	0.35

Results:

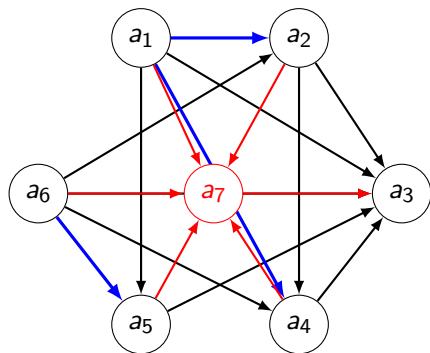
SOMETHEE I	$\{a_1, a_6\} \succ \{a_2, a_5\} \succ a_4 \succ a_3$
EXPERIMENT 1	$\{a_1, a_6\} \succ \{a_2, a_5\} \succ a_4 \succ a_7 \succ a_3$

SOMETHEE II	$a_1 \succ a_6 \succ a_2 \succ a_5 \succ a_4 \succ a_3$
EXPERIMENT 1	$a_6 \succ a_1 \succ a_2 \succ a_5 \succ a_4 \succ a_7 \succ a_3$

# At the level of the graph



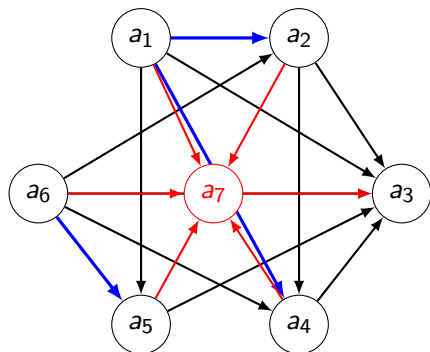
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## Experiment 1

SOMETHEE I

# At the level of the graph

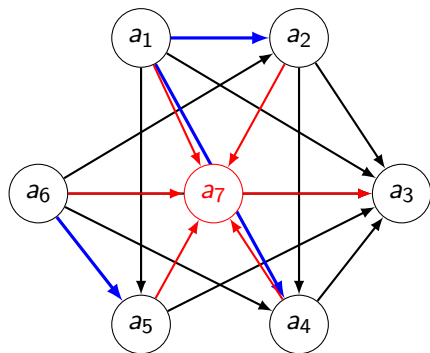


## Experiment 1

SOMETHEE I

Supplementary arcs between  
elements of  $\mathcal{A}$

# At the level of the graph



## Experiment 1

SOMETHEE I

Supplementary arcs between elements of  $\mathcal{A}$

SOMETHEE II

Change of direction of arcs between elements of  $\mathcal{A}$

## Experiment 2

Consider the supplementary alternative:

$\text{EXP}_2$	$f_1$	$f_2$	$f_3$	$f_4$
$a_7$	1	2.5	3	0.35

## Experiment 2

Consider the supplementary alternative:

EXP <sub>2</sub>	$f_1$	$f_2$	$f_3$	$f_4$
$a_7$	1	2.5	3	0.35

	Min	Min	Min	Max
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Results:

SOMETHEE I	$\{a_1, a_6\} \succ \{a_2, a_5\} \succ a_4 \succ a_3$
EXPERIMENT 2	$\{a_1, a_4, a_6\} \succ \{a_2, a_5\} \succ a_7 \succ a_3$

SOMETHEE II	$a_1 \succ a_6 \succ a_2 \succ a_5 \succ a_4 \succ a_3$
EXPERIMENT 2	$a_1 \succ a_6 \succ a_5 \succ a_4 \succ a_2 \succ a_7 \succ a_3$

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Consider the supplementary alternative:

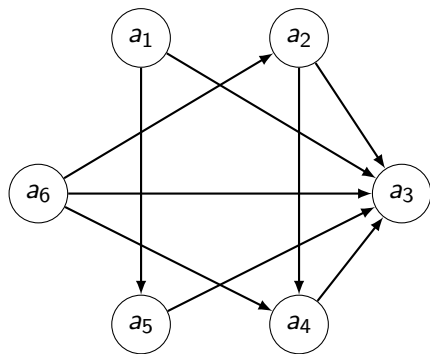
EXP <sub>2</sub>	f <sub>1</sub>	f <sub>2</sub>	f <sub>3</sub>	f <sub>4</sub>
a <sub>7</sub>	1	2.5	3	0.35

Results:

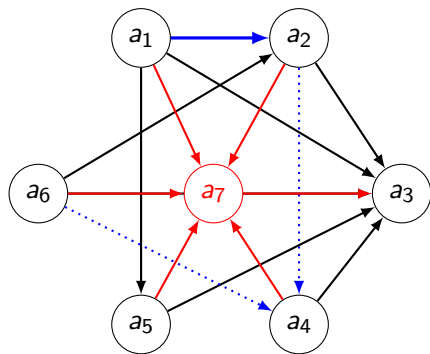
SOMETHEE I	$\{a_1, a_6\} \succ \{a_2, a_5\} \succ a_4 \succ a_3$
EXPERIMENT 2	$\{a_1, \mathbf{a_4}, a_6\} \succ \{\mathbf{a_2}, \mathbf{a_5}\} \succ a_7 \succ a_3$

SOMETHEE II	$a_1 \succ a_6 \succ a_2 \succ a_5 \succ a_4 \succ a_3$
EXPERIMENT 2	$a_1 \succ a_6 \succ \mathbf{a_5} \succ \mathbf{a_4} \succ \mathbf{a_2} \succ a_7 \succ a_3$

# At the level of the graph (bis)



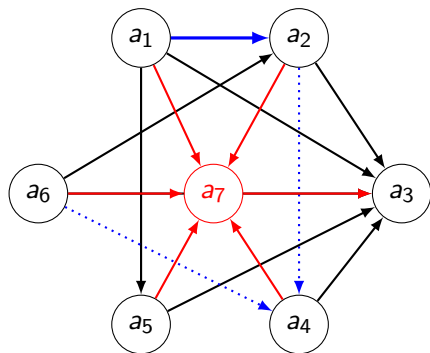
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## Experiment 2

SOMETHEE I

# At the level of the graph (bis)

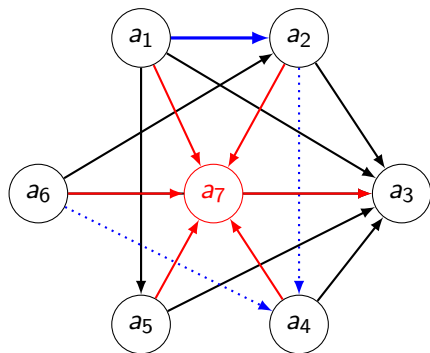


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SOMETHEE I

Supplementary arcs between  
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# At the level of the graph (bis)



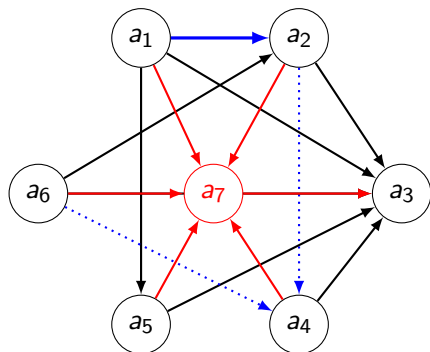
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SOMETHEE I

Supplementary arcs between  
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Deletion of arcs between  
elements of  $\mathcal{A}$

# At the level of the graph (bis)



## Experiment 2

SOMETHEE I

Supplementary arcs between elements of  $\mathcal{A}$

Deletion of arcs between elements of  $\mathcal{A}$

SOMETHEE II

Change of direction of arcs between elements of  $\mathcal{A}$

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Consider the supplementary alternative:

EXP <sub>3</sub>	$f_1$	$f_2$	$f_3$	$f_4$
$a_7$	1	2.5	1	0.35



## Experiment 3

Consider the supplementary alternative:

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$a_7$	1	2.5	1	0.35

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Results:

SOMETHEE I	$\{a_1, a_6\} \succ \{a_2, a_5\} \succ a_4 \succ a_3$
EXPERIMENT 3	$\{a_1, a_6\} \succ \{a_4, a_5\} \succ a_2 \succ a_3 \succ a_7$

SOMETHEE II	$a_1 \succ a_6 \succ a_2 \succ a_5 \succ a_4 \succ a_3$
EXPERIMENT 3	$a_6 \succ a_1 \succ a_5 \succ a_4 \succ a_2 \succ a_3 \succ a_7$

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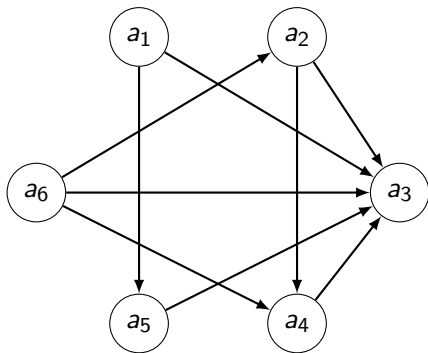
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Results:

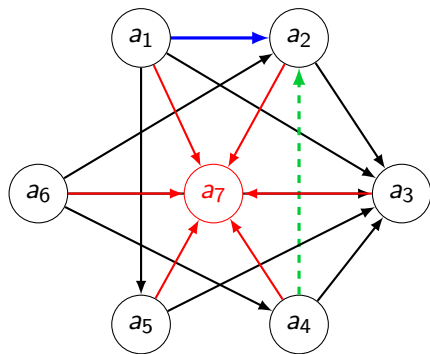
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# At the level of the graph (tris)



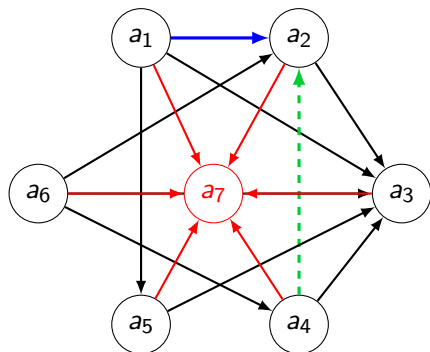
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## Experiment 3

SOMETHEE I

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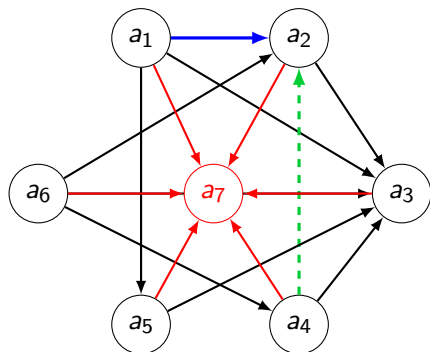


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Direction of arcs between elements of  $\mathcal{A}$  reversed

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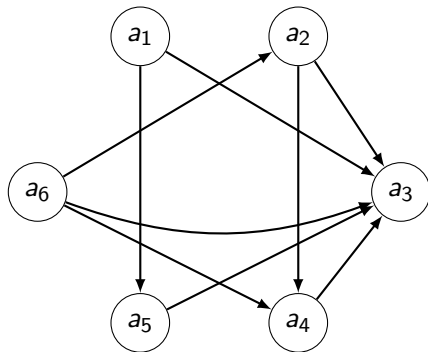
SOMETHEE I

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SOMETHEE II

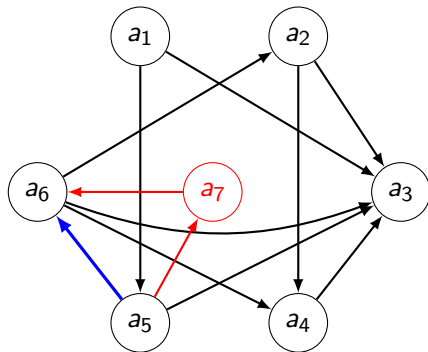
Change of direction of arcs between elements of  $\mathcal{A}$

# Something extra





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Resolve incomparability

Some food ...

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- Any outranking MCDM will be subject to type 1
- To real-life DM's type 1 can easily be explained (and they might even accept it ...)
- Type 2 should be avoided at any cost
- If methods are subject to rank reversals of type 2, users should be extremely careful!

# Conclusion

## Conjecture

The dependence of an outranking relation, based on pairwise comparisons, on the presence of a 3<sup>rd</sup> alternative is the cause of rank reversal of type 2.

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## Corollary

For any outranking MCDM, of which the outranking relation satisfies the conditions of the conjecture, it is always possible to show the existence of a rank reversal of type 2.