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Soliton Equations and Simple Combinatorics

Franklin Lambert · Johan Springael

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Abstract A systematic, elementary and pedagogical approach to a class of soliton equations, and to their spectral formulation, is presented. This approach, based on the use of exponential polynomials, follows naturally from a comparison of some simple results for two representatives of the class: the KdV- and the Boussinesq-equation.

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1 Introduction

Soliton theory originated in the discovery that a classical nonlinear wave equation, introduced by Korteweg and de Vries (KdV) for the description of long wave propagation on shallow water [24], is intimately connected with a linear Schrödinger-like eigenvalue problem [12].

In particular, it was realized that the KdV-equation:

$$\mathrm{KdV}(u) \equiv u_t - u_{3x} - 6uu_x = 0, \quad u_{px} = \frac{\partial^p u}{\partial x^p}, \tag{1}$$

coincides with the compatibility condition on the following pair of linear differential equations [26]:

$$L_2(u)\phi = \lambda\phi, \qquad L_2(u) = \partial_x^2 + u, \tag{2}$$

$$\phi_t = B_3(u)\phi, \qquad B_3(u) = 4\partial_x^3 + 6u\partial_x + 3u_x, \tag{3}$$

F. Lambert
Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussels, Belgium
e-mail: flambert@vub.ac.be

J. Springael (✉)
Departement Milieu, Technologie en Technologiemanagement, Universiteit Antwerpen, Prinsstraat 13, 2000 Antwerp, Belgium
e-mail: johan.springael@ua.ac.be

and that it may be reformulated as an operator equation, involving the commutator of the corresponding "Lax pair" (L_2, B_3):

$$[\partial_t - B_3, L_2] = 0. \tag{4}$$

The disclosure of this remarkable connection was triggered by the observation, by Miura [29], that KdV admits solutions u, which are produced by solutions σ to a "modified" version of KdV:

$$\text{mKdV}(\sigma) \equiv \sigma_t - \sigma_{3x} + 6\sigma^2\sigma_x = 0, \tag{5}$$

by means of the logarithmically linearizable transformation:

$$u = -\sigma_x - \sigma^2. \tag{6}$$

The Lax formulation (2, 3) of (1) led to a spectral interpretation of its sech squared soliton solutions (k and η are arbitrary parameters):

$$u_{\text{sol}} = \frac{k^2}{2}\text{sech}^2\left(\frac{\theta}{2}\right), \quad \theta = kx + k^3t + \eta, \tag{7}$$

and opened the way to many other findings about KdV. These include:

- the existence of infinitely many symmetries: (1) belongs to an infinite family (hierarchy) of nonlinear partial differential equations (NLPDE's), with similar properties (the KdV hierarchy);
- the existence of exact "N-soliton" solutions to (1), and to the other members of the KdV hierarchy, which describe particle-like collisions between an arbitrary number N of sech squared pulses of type (7);
- the existence of a Bäcklund transformation (BT) which relates different solutions of a "potential" version of (1);
- the possibility of solving initial value problems for (1), by means of an "inverse scattering transformation" method.

In short, it was recognized that KdV—a characteristic equation governing weakly nonlinear long waves of a quite general type—is the prototype of a "completely integrable" system [36] with infinitely many degrees of freedom.

Among other such systems represented by a NLPDE in $1+1$ dimensions, with sech squared soliton solutions and with an associated linear eigenvalue problem, there is the "good" Boussinesq or "nonlinear beam" equation [28, 34]:

$$\text{Bq}_\alpha(u) \equiv \pm 3u_{2t} - \alpha u_{2x} + (u_{2x} + 3u^2)_{2x} = 0, \quad \alpha > 0. \tag{8}$$

Its integrability properties are linked to those of the more basic equation [4]:

$$\text{Bq}(u) \equiv 3u_{2t} + (u_{2x} + 3u^2)_{2x} = 0, \tag{9}$$

to which (8) is related by the "shift": $u \to u - \frac{\alpha}{6}$.

These properties are well known [3], as are the links between KdV and Bq, as reductions of a more general three dimensional soliton equation—the KP equation [23]—which belongs to a fundamental integrable hierarchy (the KP hierarchy).

Yet, in spite of the extensive studies devoted to soliton systems in general, and to the above two representatives in particular, it seems in retrospect that only limited attention has been paid to the following basic questions:

(i) Can the pivotal spectral formulations of KdV, Bq and other similar soliton equations, be derived straight away from these NLPDE's, without relying on clever guesswork?
(ii) Are there specific elements in (1) and (9) suggestive of the fact that KdV and Bq are to be regarded as dimensional reductions of a more fundamental NLPDE in 3 dimensions?
(iii) Can one generate families of sech squared soliton equations, together with their linear spectral formulation, by means of an algorithmic procedure, capable of producing KP, as well as other members of the KP hierarchy and of closely related hierarchies?

Partial answers to these questions can be found in the abundant literature on the algebraic and analytical techniques that have been developed in soliton theory [9, 10, 27], and more specifically in papers on Hirota's "direct" method [17]. Unfortunately, this method according to which the original NLPDE is to be replaced by more tractable bilinear equations for new dependent variables [18], and which, in the hands of experts has proved particularly powerful, is not as *direct* as one may wish. There is, as yet, no general rule for the selection of the Hirota variables,[1] nor for the choice and application of some essential formulas (such as the s.c. "exchange" formulas). On the other hand, there are recent indications [25] that the above questions might be answered straight away by means of simple considerations, such as dimensional analysis and elementary combinatorics. This possibility, and the fact that direct insight into the linear background of soliton equations may be gained much more easily (i.e. without the introduction of other differential operators than the most ordinary ones) seem to have been overlooked: a shortcoming all the more surprising that the tools needed for this task are available since many years.

The aim of the present paper is to fill this gap, by showing that major integrability features of a significant set of soliton equations can be disclosed by means of a direct approach, starting with an elementary analysis of KdV, mKdV and Bq, and ending with a comprehensive picture of the linear multidimensional spaces to which these NLPDE's belong.

The various sections of the paper are organized so as to serve another purpose: to show that useful but seemingly artificial tricks, of current use in soliton studies can either be circumvented, or justified step by step.

Thus, in order to introduce key elements of our analysis (such as the notion of dimension or weight of each variable) and to motivate the extensive use of exponential polynomials (as a way of expressing the particular balance between dispersion and nonlinearity in soliton equations), we start our discussion with a characterization of some linearizable NLPDE's with solitary wave solutions, but without solitons (Sect. 2).

In Sect. 3 we clarify the link between KdV and the Lax system, by showing that a simple combinatorial rule—the one which governs the structure of logarithmically linearizable NLPDE's—applies also to the primary version of a system constituted by the Miura transformation and the mKdV equation.

In Sect. 4 the linear system (2, 3) is shown to be obtainable from KdV itself, with the help of exponential polynomials of a *binary* type ($\mathcal{Y}$-polynomials): it means that the Lax system is nothing else than a linearized version of two "$\mathcal{Y}$-constraints" into which a simple invariance condition on the primary KdV expression can be decoupled.

In Sect. 5 it is found that the $\mathcal{Y}$-constraints, which produce the Lax system, yield also the (auto) Bäcklund transformation (BT) for KdV, as well as the crucial Miura link between KdV and mKdV.

[1] In some cases it was found [13, 16] that application of Painlevé analysis to the NLPDE may provide guidance in the choice of appropriate dependent variables which allow a "bilinearization" of the equation. Sometimes independent "auxiliary" variables must be introduced as well.

In Sects. 6 and 7 we show that similar results (i.e. spectral formulation, BT and Miura-like transformation) can be obtained for Bq by application of the same *direct* procedure.

In Sect. 8 we discuss the relevance of "$\mathcal{P}$-conditions" (i.e. reduced $\mathcal{Y}$-constraints in which one variable is set equal to zero). A fifth order alternative to KdV (the SK equation) is constructed. It is pointed out that $1+1$ dimensional $\mathcal{P}$-conditions of increasing order cannot be used, as such, for the construction of families of sech squared soliton equations.

In Sect. 9, a more adequate procedure for the construction of such families is suggested by a detailed comparison of the results obtained for KdV and Bq. It makes use of a multi-dimensional extension of the bases of $\mathcal{P}$- and $\mathcal{Y}$-polynomials considered so far: KdV and Bq are found to correspond to *complementary* dimensional reductions of a 3 dimensional NLPDE, the primary version of which is identified as a basic $\mathcal{P}$-condition of weight 4 (KP_4). This $\mathcal{P}$-condition is a potential version of the compatibility condition on a fundamental pair of $\mathcal{Y}$-constraints (mKP_2 and mKP_3).

In Sect. 10 we construct an alternative to KP_4 of weight 6 (BKP_6) by means of another fundamental pair of $\mathcal{Y}$-constraints (DKP_3, DKP_5). We show that BKP_6 produces two $1+1$ dimensional soliton equations (SK and Ramani [30]) in the form of complementary reductions, in close analogy with KdV and Bq.

In Sect. 11 the basic character of KP_4 is confirmed with the identification of a family of $\mathcal{Y}$-constraints of weight 4 (mKP_4 family), all members of which are compatible with mKP_2 and mKP_3 under KP_4 and a "companion" of weight 5 (KP_5).

Section 12 is devoted to the construction of a higher member of the potential Bq family, which arises as a "t_3-reduction" of the pair (KP_4, KP_5). A spectral formulation of this higher member is obtained by application of the direct procedure outlined in Sect. 4.

In Sect. 13 we continue the construction process initiated in Sect. 11, so as to obtain a weight 6 member of the primary KP family (KP_6). We then show that a higher member of the KdV family (the Lax equation) can be identified as a t_2-reduction of the pair (KP_4, KP_6). We derive a spectral formulation of this higher KdV equation, and find that it admits a one parameter family of equivalent $1+2$ dimensional "$\mathcal{P}$-representations".

The method of "$\mathcal{P}$-decomposition" (suggested by the results of Sects. 12 and 13) is applied in Sect. 14 to a parameter family of shallow water wave equations. It produces two genuine soliton equations (HS and AKNS) [1, 19], both of which are found to belong to a known family.

2 Logarithmically Linearizable NLPDE's

Linear partial differential equations easily admit families of exact solutions with an arbitrary number of free parameters. This follows from the superposition principle, according to which any linear combination of particular solutions is a solution as well. In the case of an NLPDE (to which the linear superposition does not apply) the occurrence of such parameter families of exact solutions is harder to understand. It suggests the existence of a special link between the NLPDE and a linear equation (or system of linear equations).

2.1 The Burgers Equation

The most direct link one may think of is the existence of a linearizing transformation. This is the case for a family of NLPDE's—the Burgers-Hopf (BH) family [11]—the simplest member of which is the Burgers equation [5]:

$$\mathrm{B}(u) \equiv u_t - u_{2x} - uu_x = 0. \tag{10}$$

Equation (10) is known to be linked to the linear diffusion equation

$$\mathrm{B}_0(\phi) \equiv \phi_t - \phi_{2x} = 0, \tag{11}$$

by the Hopf-Cole transformation [7, 21]:

$$u = 2\partial_x \ln\phi. \tag{12}$$

It therefore admits solitary wave solutions (shock waves):

$$u_{\mathrm{sol}} = 2\partial_x \ln(1+\exp\theta), \quad \theta = kx + k^2 t + \eta, \tag{13}$$

as well as more general "N step" solutions (k_i and η_i are constants):

$$u_N = 2\partial_x \ln\left(1+\sum_{i=1}^{N}\exp\theta_i\right), \quad \theta_i = k_i x + k_i^2 t + \eta_i. \tag{14}$$

2.2 Exponential Polynomials and Y-Conditions

Is there an ingredient in (10), and in the other members of the BH family, which accounts for the Hopf-Cole linearizability of these NLPDE's?

To find out, we consider the following set of polynomials (a two dimensional generalization of Bell's exponential polynomials [2, 8]):

$$\{Y_{mx,nt}(v) = e^{-v}\partial_x^m \partial_t^n e^v,\ m,n = \text{integer or zero}\}. \tag{15}$$

They display an easily recognizable "partitional" balance between their linear and their nonlinear terms ($v_{mx,nt}$ is balanced by as many nonlinear terms as there are different partitions of $m+n$ into two or more parts, each nonlinear term being accompanied by a coefficient equal to the combinatorial weight of the corresponding partition):

$$\begin{aligned} &Y_x(v) = v_x, \qquad Y_{2x}(v) = v_{2x} + v_x^2, \qquad Y_{x,t}(v) = v_{x,t} + v_x v_t,\\ &Y_{3x}(v) = v_{3x} + 3v_x v_{2x} + v_x^3, \quad \ldots. \end{aligned} \tag{16}$$

In the following we call "Y-condition" any NLPDE in the form of a linear combination of Y-polynomials set equal to zero.

On account of the definition (15), it is clear that each Y-condition on v can be transformed into a linear differential equation (with constant coefficients) by the transformation $v = \ln\phi$.

We now notice that the Burgers equation (10) remains invariant under the scale transformation:

$$x \to \alpha x, \qquad t \to \alpha^2 t, \qquad u \to \alpha^{-1} u. \tag{17}$$

Thus, setting $u = cv_x$, we introduce a dimensionless *potential* variable v (c = dimensionless parameter) and find that (10) can be derived from a "potential" Burgers equation:

$$v_t - \left(v_{2x} + \frac{c}{2}v_x^2\right) = 0, \tag{18}$$

which happens to be expressible as a Y-condition, upon an appropriate choice for c ($c = 2$).

Hence, it is found that the Hopf-Cole linearizability of Burgers is due to a special balance between its linear and its nonlinear parts, the *partitional* nature of which becomes explicit when the NLPDE is expressed in terms of a *primary* variable v, introduced by setting $u = 2v_x$. The potential Burgers equation is then seen to be the x-derivative of the following Y-condition:

$$E_{\text{Burgers}}(v) \equiv Y_t(v) - Y_{2x}(v) = 0. \tag{19}$$

We conclude that Hopf-Cole-like linearizability of a given NLPDE is easy to detect: it suffices to look for a "primary" version of that NLPDE in the form of a Y-condition on a dimensionless potential alternative to the original dependent variable.

An example is the Sharma-Tasso-Olver equation [15]:

$$u_t + u_{3x} + 3(uu_{2x} + u_x^2 + u^2 u_x) = 0. \tag{20}$$

Setting $u = cv_x$, on account of the dimension of u ($\dim u = -1$), we find that, at $c = 1$, (20) can be derived from the logarithmically linearizable primary NLPDE:

$$v_t + v_{3x} + 3v_x v_{2x} + v_x^3 \equiv Y_t(v) + Y_{3x}(v) = 0. \tag{21}$$

3 Spectral Formulation of KdV

Soliton equations cannot be linearized by a mere change of dependent variable. And yet, they are known to be closely linked to a system of linear differential equations. Let us see what we can learn about this link by examining the Miura connection between KdV and mKdV at a "primary" level.

3.1 Primary Versions of KdV and mKdV

Setting $\sigma = v_x$ and $u = q_{2x}$, on account of the dimension of the variables,[2] we find that mKdV and KdV can both be derived from a potential NLPDE for a dimensionless alternative to the original dependent variable:

$$E_{\text{mKdV}}(v) \equiv v_t - v_{3x} + 2v_x^3 = 0, \tag{22}$$

$$E_{\text{KdV}}(q) \equiv q_{x,t} - q_{4x} - 3q_{2x}^2 = 0. \tag{23}$$

Both "primary" NLPDE's are homogeneous, of a definite weight (if we define the weight of a term as being minus its dimension: E_{mKdV} is of weight 3, E_{KdV} is of weight 4). They also exhibit a striking "parity":

- E_{mKdV} contains only *odd* order derivatives and can easily be expressed in terms of Y-polynomials (its degree still matches its order in x):

$$E_{\text{mKdV}}(v) \equiv Y_t(v) - Y_{3x}(v) + 3Y_x(v)Y_{2x}(v) = 0; \tag{24}$$

[2] mKdV and KdV are invariant under the scale transformation:

$$x \to \alpha x, \qquad t \to \alpha^3 t, \qquad \sigma \to \alpha^{-1}\sigma, \qquad u \to \alpha^{-2} u.$$

- E_{KdV} involves only *even* order derivatives and can be expressed as a linear combination of two "reduced" Y-polynomials of even order, in which all factors with an odd number of derivatives have been set equal to zero:

$$E_{\mathrm{KdV}}(q) = \mathcal{P}_{x,t}(q) - \mathcal{P}_{4x}(q), \tag{25}$$

with

$$\mathcal{P}_{mx,nt}(q) = Y_{mx,nt}(q)|_{q_{rx,st}=0 \text{ if } r+s=\text{odd}}. \tag{26}$$

3.2 Miura's Transformation and the Lax System

As the connecting Miura transformation (6) is expressed in terms of the primary variables v and q it is found to contain odd as well as even order derivatives, and to be expressible as a homogeneous and logarithmically linearizable constraint on v and q:

$$\mathrm{Miura}(v,q) \equiv v_{2x} + q_{2x} + v_x^2 \equiv Y_{2x}(v) + q_{2x} = 0. \tag{27}$$

This constraint (of weight 2) happens to transform $E_{\mathrm{mKdV}}(v)$ into another homogeneous logarithmically linearizable constraint (of weight 3):

$$\mathrm{Miura}(v,q) = 0, \qquad Y_{2x}(v) + q_{2x} = 0, \tag{28}$$

$$\Longleftrightarrow$$

$$E_{\mathrm{mKdV}}(v) = 0 \qquad Y_t(v) - Y_{3x}(v) - 3q_{2x}Y_x(v) = 0. \tag{29}$$

Thus, as the primary mKdV equation is taken in association with the Miura constraint, the two equations constitute a system which is equivalent with the following linear system for $\phi = \exp v$:

$$L_2(q_{2x})\phi = 0, \qquad L_2(q_{2x}) = \partial_x^2 + q_{2x}, \tag{30}$$

$$\phi_t - \phi_{3x} - 3q_{2x}\phi_x = 0. \tag{31}$$

Compatibility of the Miura constraint with the primary mKdV equation is now subject to the integrability of this linear system, i.e. to a condition on q_{2x} which is nothing else than the KdV equation,[3] written in terms of the primary variable q:

$$\partial_x E_{\mathrm{KdV}}(q) \equiv [\mathcal{P}_{x,t}(q) - \mathcal{P}_{4x}(q)]_x = 0. \tag{32}$$

Keeping in mind that $q_{2x} = u$, it follows from the invariance of KdV under the "shift" $u \to u + \lambda$, combined with the Galilean transformation $x \to x - 6\lambda t$, that (32) can also be regarded as the compatibility condition on the following parameter dependent pair of logarithmically linearizable constraints:

$$Y_{2x}(v) + q_{2x} = \lambda, \tag{33}$$

$$Y_t(v) - Y_{3x}(v) - 3(q_{2x} + \lambda)Y_x(v) = 0. \tag{34}$$

[3] Miura's observation implies that (32) is satisfied if v and q satisfy the constraints (28, 29).

The latter are mapped, by the linearizing transformation $v = \ln\phi$, onto a λ-dependent alternative to system (30, 31), which now comprises the familiar eigenvalue equation associated with the operator L_2:

$$L_2(q_{2x})\phi = \lambda\phi, \tag{35}$$

$$\phi_t - \phi_{3x} - 3(q_{2x} + \lambda)\phi_x = 0. \tag{36}$$

System (35, 36) takes the form of the Lax system (2, 3), after elimination of the λ-dependent term from (36).

3.3 $\mathcal{P}$-Conditions

A noteworthy point about KdV is its linearity with respect to a "basis" of $\mathcal{P}$-polynomials: $E_{\mathrm{KdV}}(q)$ is a linear combination of such polynomials and displays an easily recognizable balance between its linear and its nonlinear terms (each linear term contains an even number of derivatives, and is balanced by as many different nonlinear ones as there are different partitions of that number into even parts).

In the following we shall call "$\mathcal{P}$-condition" any condition on a dimensionless variable q, which corresponds to a linear combination of $\mathcal{P}$-polynomials set equal to zero.

The primary version (23) of KdV can then be viewed as a homogeneous $\mathcal{P}$-condition of weight 4.

Inspection of (8) and (9) shows that in the Bq (or Bq_α) case a dimensionless (weightless) alternative to the dependent variable u can equally be introduced by setting $u = q_{2x}$. This substitution leads (after two integrations with respect to x) to primary versions of Bq and Bq_α in the form of $\mathcal{P}$-conditions of weight 4 (notice that $\dim\alpha = -2$):

$$E_{\mathrm{Bq},\alpha}(q) \equiv 3\mathcal{P}_{2t}(q) - \alpha\mathcal{P}_{2x}(q) + \mathcal{P}_{4x}(q) = 0, \tag{37}$$

$$E_{\mathrm{Bq}}(q) \equiv 3\mathcal{P}_{2t}(q) + \mathcal{P}_{4x}(q) = 0. \tag{38}$$

It follows that KdV and Bq_α are sech squared soliton equations which can both be derived from a primary $\mathcal{P}$-condition (we shall come back to this point in Sect. 8).

4 Disclosure of the Lax System

We retrieved the Lax system for KdV from Miura's connection between KdV and mKdV. We shall now show that KdV's spectral formulation can also be obtained without the a priori knowledge of the Miura transformation: the Lax system (2, 3) can be derived from $E_{\mathrm{KdV}}(q)$ itself.

4.1 $\mathcal{Y}$-Polynomials and $\mathcal{Y}$-Constraints

To find a way leading directly from $E_{\mathrm{KdV}}(q)$ to the Lax system, it suffices to take a close look at the constraints (27, 33) and (34). Each of them can also be viewed as a constraint which relates odd order derivatives of v to an even order derivative of the *new* variable $w = v + q$:

$$\mathrm{Miura}(v, q) = 0 \iff w_{2x} + v_x^2 = 0, \tag{39}$$

$$Y_{2x}(v) + q_{2x} = \lambda \iff w_{2x} + v_x^2 = \lambda, \tag{40}$$

$$Y_t(v) - Y_{3x}(v) - 3(q_{2x} + \lambda)Y_x(v) = 0 \iff v_t - v_{3x} - 3v_x w_{2x} - v_x^3 - 3\lambda v_x = 0. \tag{41}$$

This observation suggests the introduction of *binary* exponential polynomials, i.e. Y-polynomials in which all *even* order derivatives of v have been re-labelled as corresponding derivatives of w:

$$\mathcal{Y}_{mx,nt}(v,w) = Y_{mx,nt}(v)|_{v_{rx,st}=w_{rx,st} \text{ if } r+s=\text{even}}. \tag{42}$$

We then find that each of the above constraints (39–41) can be rewritten as a constraint on v and w which is linear with respect to a basis of $\mathcal{Y}$-polynomials ($\mathcal{Y}$-basis).

In the following, we call "$\mathcal{Y}$-constraint" any constraint on a pair of dimensionless variables v and w, which can be expressed as a linear combination of $\mathcal{Y}$-polynomials set equal to zero.

The λ-dependent nonlinear constraints (33, 34), which produced the Lax system for $\phi = \exp v$, can then be viewed as a pair of $\mathcal{Y}$-constraints (of weight 2 and 3):

$$\mathcal{Y}_{2x}(v,w) = \lambda, \tag{43}$$

$$\mathcal{Y}_t(v) - \mathcal{Y}_{3x}(v,w) - 3\lambda\mathcal{Y}_x(v) = 0, \tag{44}$$

the compatibility of which is subject to a nonlinear condition on $q = w - v$. This condition is linear with respect to the "$\mathcal{P}$-basis" (constituted by *even* order $\mathcal{Y}$-polynomials in which v is set equal to zero). Notice that *odd* order $\mathcal{Y}$-polynomials vanish identically at $v = 0$, and that:

$$\mathcal{Y}_{mx,nt}(0,q) = \mathcal{P}_{mx,nt}(q), \quad m+n = \text{even}. \tag{45}$$

4.2 A Darboux Invariance Property of KdV

Since solutions to mKdV appear in pairs $(\sigma, -\sigma)$, on account of an obvious symmetry of (5), these should be mapped by the Miura transformation (6) onto pairs $(u, \tilde{u})$ of solutions of KdV:

$$u = -\sigma_x - \sigma^2, \qquad \tilde{u} = \sigma_x - \sigma^2. \tag{46}$$

Thus, to the symmetry of mKdV with respect to a change of sign in the dependent variable, there must correspond a symmetry of KdV under the map:

$$u \to \tilde{u} = u + 2\sigma_x, \tag{47}$$

in which u represents a solution of KdV, and $\sigma = \partial_x \ln\phi$ is determined by a solution of the Lax system (2, 3), taken with that u.

To show the invariance of $E_{\text{KdV}}(q)$ under a corresponding "Darboux" map:

$$q \to \tilde{q} = q + 2\ln\phi, \tag{48}$$

it suffices to verify that the invariance condition:

$$E_{\text{KdV}}(w+v) - E_{\text{KdV}}(w-v) \equiv 2(v_{x,t} - v_{4x} - 6v_{2x}w_{2x}) = 0 \tag{49}$$

is satisfied whenever v and w obey the $\mathcal{Y}$-constraints (43, 44). This can be done straight away by substitution (after differentiation of (44) with respect to x).

It follows that the Lax system (2, 3) is a linearized version of a "KdV compatible" parameter dependent pair of $\mathcal{Y}$-constraints into which the invariance condition (49) can be decoupled, the linearizing transformation being:

$$w = v + q, \quad v = \ln\phi. \tag{50}$$

This observation opens the way to a direct disclosure of the Lax formulation of KdV (and of similar spectral formulations of other sech squared soliton equations). A straightforward procedure, based on a systematic use of $\mathcal{Y}$-constraints, can be applied for this purpose. It relies on the following remarkable property.

4.3 Key-Property of $\mathcal{Y}$-Constraints

It is easy to check that any constraint of the form:

$$\sum_j c_j \mathcal{Y}_{m_j x, n_j t}(v, w) = 0, \quad c_j = \text{constant}, \tag{51}$$

is mapped by the substitutions (50) onto a linear differential equation for ϕ of the form:

$$\sum_j c_j \sum_{r_j=0}^{m_j} \sum_{s_j=0}^{n_j} \binom{m_j}{r_j} \binom{n_j}{s_j} \mathcal{P}_{r_j x, s_j t}(q) \phi_{(m_j - r_j)x, (n_j - s_j)t} = 0. \tag{52}$$

This follows from the logarithmic linearizability of Y-polynomials, and from the definition of $\mathcal{Y}_{mx,nt}(v, v+q)$, on account of which (application of the Leibniz rule):

$$\mathcal{Y}_{mx,nt}(v, v+q) = \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r} \binom{n}{s} \mathcal{Y}_{rx,st}(0, q) \mathcal{Y}_{(m-r)x,(n-s)t}(v, v), \tag{53}$$

with

$$\mathcal{Y}_{mx,nt}(v, v) = Y_{mx,nt}(v). \tag{54}$$

4.4 Direct Procedure

It is now clear that the Lax formulation of KdV can be obtained from the primary expression $E_{\text{KdV}}(q)$, by proceeding as follows:

(i) reformulation of the invariance condition (49) in terms of $\mathcal{Y}$-polynomials and their first order derivatives[4] ($\mathcal{Y}_{px,qt}$ stands for $\mathcal{Y}_{px,qt}(v, w)$):

$$E_{\text{KdV}}(w+v) - E_{\text{KdV}}(w-v) \equiv 2\partial_x(\mathcal{Y}_t - \mathcal{Y}_{3x}) + 6W[\mathcal{Y}_x, \mathcal{Y}_{2x}] = 0, \tag{55}$$

where $W[f, g]$ denotes the Wronskian $fg_x - f_x g$.

(ii) identification of an auxiliary $\mathcal{Y}$-constraint of lowest possible weight, relating v to w, which reduces condition (55) to the x-derivative of another $\mathcal{Y}$-constraint. The simplest candidate is the homogeneous $\mathcal{Y}$-constraint of weight 2: $\mathcal{Y}_{2x}(v, w) = 0$. It reduces condition (55) to the x-derivative of a homogeneous $\mathcal{Y}$-constraint of weight 3:

$$\mathcal{Y}_t(v) - \mathcal{Y}_{3x}(v, w) = 0. \tag{56}$$

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[4] It suffices to use the following standard substitutions:

$$v_{x,t} = \partial_x \mathcal{Y}_t, \qquad v_{2x} = \partial_x \mathcal{Y}_x, \qquad w_{2x} = \mathcal{Y}_{2x} - \mathcal{Y}_x^2,$$
$$v_{4x} = \partial_x(\mathcal{Y}_{3x} - 3\mathcal{Y}_x \mathcal{Y}_{2x} + 2\mathcal{Y}_x^3).$$

(iii) application of the substitutions (50) in order to transform the homogeneous "$\mathcal{Y}$-system":

$$\mathcal{Y}_{2x}(v,w)=0, \tag{57}$$

$$\mathcal{Y}_t(v)-\mathcal{Y}_{3x}(v,w)=0 \tag{58}$$

into a corresponding linear differential system for ϕ:

$$\phi_{2x}+q_{2x}\phi=0, \tag{59}$$

$$\phi_t-\phi_{3x}-3q_{2x}\phi_x=0. \tag{60}$$

(iv) verification that the integrability of system (59, 60) is subject to a condition on q which is satisfied if $E_{\mathrm{KdV}}(q)=0$.

(v) transformation of the $\mathcal{Y}$-system (57, 58) into a parameter dependent alternative, which comprises the nonlinear equivalent to a t-independent linear eigenvalue equation for ϕ (the compatibility of the parameter dependent $\mathcal{Y}$-constraints should remain independent of the parameter). The parameter dependent alternative[5] to system (57, 58) is system (43, 44).

(vi) application of the substitutions (50), in order to transform system (43, 44) into a corresponding linear system for ϕ, and elimination of the parameter from the "t-dependent" member of this system.

To summarize, one may say that the Lax system for KdV can be obtained from $E_{\mathrm{KdV}}(q)$, by means of an appropriate parameter dependent "$\mathcal{Y}$-basis" decomposition of the corresponding invariance condition (55).

5 Miura and Bäcklund Transformations

Let us consider step (ii) of the above procedure in some more detail. The simplest auxiliary $\mathcal{Y}$-constraint which produces a suitable $\mathcal{Y}$-basis decomposition of condition (55) is the weight 2 constraint (57). Regarded as a constraint on v and $q=w-v$, it is nothing else than the Miura constraint:

$$\mathcal{Y}_{2x}(v,v+q)\equiv v_{2x}+q_{2x}+v_x^2=0. \tag{61}$$

Its introduction transforms condition (55) into a homogeneous system:

$$v_{2x}+q_{2x}+v_x^2=0, \tag{62}$$

$$v_{x,t}-v_{4x}-6v_{2x}(v_{2x}+q_{2x})=0, \tag{63}$$

which (after elimination of q_{2x} from (63)) produces the mKdV condition on v. It means that the crucial Miura connection between KdV and mKdV is a result which appears naturally at the second step of the direct procedure.

[5] A spectral parameter can be introduced in two ways: either as a "decoupling" constant in the auxiliary $\mathcal{Y}$-constraint, or as an integration constant in the other one.

On the other hand, it is found at step (v) that the introduction of the auxiliary constraint (43) transforms the condition (55) into a coupled system for v and w:

$$w_{2x} + v_x^2 = \lambda, \tag{64}$$

$$v_{x,t} - v_{4x} - 6v_{2x}w_{2x} = 0, \tag{65}$$

which may be rewritten in terms of $\tilde{V} - V = 2v_x$ and $\tilde{V} + V = 2w_x$:

$$(\tilde{V} + V)_x + \frac{1}{2}(\tilde{V} - V)^2 = \lambda, \tag{66}$$

$$(\tilde{V} - V)_t - (\tilde{V} - V)_{3x} - 3(\tilde{V} - V)_x(\tilde{V} + V)_x = 0. \tag{67}$$

The latter is a system for $\tilde{V}$ which, by construction, is integrable if $V = q_x$ solves the primary KdV equation (23), expressed in terms of V:

$$\mathrm{PKdV}(V) \equiv V_t - V_{3x} - 3V_x^2 = 0, \tag{68}$$

and which produces another solution $\tilde{V}$ to this equation (on account of the invariance condition (55)).

This property is known as the (auto)Bäcklund property: system (66, 67) is the well known Bäcklund transformation [32] which relates a pair of solutions V and $\tilde{V}$ to the potential KdV equation (68).

6 Spectral Formulation of the PBq System

In order to test the "direct" procedure, we apply it to the primary Bq equation (38).

We first express the invariance condition on $E_{\mathrm{Bq}}(q)$ in terms of $\mathcal{Y}$-polynomials, and their first order derivatives (step (i)):

$$\begin{aligned} &E_{\mathrm{Bq}}(w+v) - E_{\mathrm{Bq}}(w-v) \\ &\quad \equiv 2(3v_{2t} + v_{4x} + 6v_{2x}w_{2x}) \\ &\quad \equiv 2\partial_x[\mathcal{Y}_{3x}(v,w)] + 6\partial_t[\mathcal{Y}_t(v)] - 6W[\mathcal{Y}_x(v), \mathcal{Y}_{2x}(v,w)] = 0. \end{aligned} \tag{69}$$

We then examine whether $E_{\mathrm{Bq}}(w+v) - E_{\mathrm{Bq}}(w-v)$ can be reduced to the x-derivative of a linear combination of $\mathcal{Y}$-polynomials of weight 3, say (the t-variable in Bq has dimension 2):

$$\mathcal{Y}_{3x}(v,w) + \alpha\mathcal{Y}_{x,t}(v,w), \quad \alpha = \text{dimensionless constant}, \tag{70}$$

by means of an appropriate (homogeneous) auxiliary $\mathcal{Y}$-constraint of weight 2:

$$\mathcal{Y}_t(v) + \beta\mathcal{Y}_{2x}(v,w) = 0, \quad \beta = \text{dimensionless constant}. \tag{71}$$

To find out, it suffices to rewrite condition (69) as:[6]

$$\begin{aligned} &\partial_x[\mathcal{Y}_{3x} + \alpha\mathcal{Y}_{x,t}] + 3\partial_t[\mathcal{Y}_t + \beta\mathcal{Y}_{2x}] - (\alpha + 3\beta)(\partial_x\mathcal{Y}_{x,t}) \\ &\quad - 3W[\mathcal{Y}_x, \beta\mathcal{Y}_t + \mathcal{Y}_{2x}] = 0, \end{aligned} \tag{72}$$

[6] We use the elementary identity: $\partial_t\mathcal{Y}_{2x} \equiv \partial_x\mathcal{Y}_{x,t} + W[\mathcal{Y}_x, \mathcal{Y}_t]$.

or as (choosing $\alpha = 3$ and $\beta = -1$):

$$\partial_x[\mathcal{Y}_{3x} + 3\mathcal{Y}_{x,t}] + 3\partial_t[\mathcal{Y}_t - \mathcal{Y}_{2x}] + 3W[\mathcal{Y}_x, \mathcal{Y}_t - \mathcal{Y}_{2x}] = 0. \tag{73}$$

Equation (73) shows that the invariance condition on $E_{\mathrm{Bq}}(q)$ can be decoupled into the following two $\mathcal{Y}$-constraints (we introduce a spectral parameter μ of weight 3, in the form of an integration constant):[7]

$$\mathcal{Y}_t(v) - \mathcal{Y}_{2x}(v, w) = 0, \tag{74}$$

$$\mathcal{Y}_{3x}(v, w) + 3\mathcal{Y}_{x,t}(v, w) = \mu. \tag{75}$$

Their compatibility is subject to the integrability of the corresponding linear system for ϕ (application of the substitutions (50), and of formula (52)):

$$\phi_t = L_2(q_{2x})\phi, \quad L_2(q_{2x}) = \partial_x^2 + q_{2x}, \tag{76}$$

$$L_3(q_{2x}, q_{x,t})\phi = \mu\phi, \quad L_3(q_{2x}, q_{x,t}) = \partial_x^3 + \frac{3}{2}q_{2x}\partial_x + \frac{3}{4}(q_{3x} + q_{x,t}), \tag{77}$$

i.e. to a condition on q which is found to coincide with the following potential version of Bq:

$$\partial_x E_{\mathrm{Bq}}(q) \equiv [3\mathcal{P}_{2t}(q) + \mathcal{P}_{4x}(q)]_x = 0. \tag{78}$$

The linear system (76, 77) provides us with a spectral formulation of Bq, known [35] as its “Lax system”: it involves a linear evolution equation governed by KdV’s second order Lax operator L_2, and a third order eigenvalue equation associated with an operator L_3, which, up to a single nondifferential term, corresponds to KdV’s Lax operator B_3 divided by a factor 4.

The presence in L_3 of two “potentials” $u = q_{2x}$ and $r = q_{x,t}$, linked by the obvious relation $u_t = r_x$, is an indication that (78) can also be represented by the following coupled system of evolution equations:

$$u_t = r_x, \tag{79}$$

$$r_t = -\frac{1}{3}u_{3x} - 2uu_x. \tag{80}$$

Indeed, to verify that these equations are the conditions that must be satisfied in order that the system:

$$\phi_t = L_2(u)\phi, \tag{81}$$

$$L_3(u, r)\phi = \mu\phi, \tag{82}$$

be integrable, it suffices to consider the commutator relation:

$$[\partial_t - L_2(u), L_3(u, r)] = \frac{3}{2}(u_t - r_x)\partial_x + \frac{3}{4}\left[(u_t - r_x)_x + \left(r_t + \frac{1}{3}u_{3x} + 2uu_x\right)\right]. \tag{83}$$

[7] In contrast with the KdV-case, one can easily verify that an auxiliary constraint of the form $\mathcal{Y}_t - \mathcal{Y}_{2x} = \lambda$ does not produce a decoupling of condition (73) into a pair of $\mathcal{Y}$-constraints of the required type (leading to an eigenvalue equation for $\phi = \exp v$, associated with a linear differential operator with respect to x) unless $\lambda = 0$.

7 Bäcklund and Miura Transformations for Bq

Let us now go back to the $\mathcal{Y}$-constraints (74, 75), into which the invariance condition on $E_{\mathrm{Bq}}(q)$ can be decoupled.

Elimination of v_t from the second constraint leads, upon differentiation with respect to x, to a coupled system for $2v_x = \tilde{V} - V$ and $2w_x = \tilde{V} + V$:

$$(\tilde{V} - V)_t = (\tilde{V} + V)_{2x} + (\tilde{V} - V)(\tilde{V} - V)_x, \tag{84}$$

$$3(\tilde{V} + V)_t = 2\mu - (\tilde{V} - V)_{2x} - 3(\tilde{V} - V)(\tilde{V} + V)_x - (\tilde{V} - V)^3, \tag{85}$$

which, by construction, is integrable when $V = q_x$ satisfies the potential Bq equation (78), expressed in terms of V:

$$\mathrm{PBq}(V) \equiv 3V_{2t} + V_{4x} + 6V_x V_{2x} = 0, \tag{86}$$

and which produces another solution $\tilde{V}$ of this same equation (auto-Bäcklund property).

System (84, 85) is known as the Bäcklund transformation [6] for Bq.

In analogy with the KdV-case, it is also found that the invariance condition on the primary expression $E_{\mathrm{Bq}}(q)$, taken together with the auxiliary $\mathcal{Y}$-condition of weight 2 (which reduces that condition to the x-derivative of a $\mathcal{Y}$-constraint of weight 3) produces a coupled system for the variables v and $q = w - v$:

$$3v_{2t} + v_{4x} + 6v_{2x}(v_{2x} + q_{2x}) = 0, \tag{87}$$

$$v_t - (v_{2x} + q_{2x}) - v_x^2 = 0, \tag{88}$$

which upon elimination of q_{2x}, yields a “modified” version of Bq [20]:

$$\mathrm{mBq}(v) \equiv 3v_{2t} + v_{4x} + 6v_{2x}v_t - 6v_x^2 v_{2x} = 0. \tag{89}$$

Solutions v to this NLPDE produce solutions u to the Bq equation (9), according to the Miura-like transformation:[8]

$$u = v_t - v_{2x} - v_x^2. \tag{90}$$

To the obvious symmetry of (89) under the transformation ($v \to -v$, $t \to -t$) there corresponds the less obvious symmetry of Bq, under the map: $u \to \tilde{u} = u + 2v_{2x}$. This symmetry is reflected, at the potential level, by the invariance property:[9]

$$\mathrm{PBq}(V + 2v_x) = \mathrm{PBq}(V), \tag{91}$$

in which V represents a solution of (86), and $v = \ln \phi$ is determined by a solution ϕ of the “Lax system” (76, 77), taken with $q_x = V$.

We conclude that KdV and PBq display a strong analogy: both equations are expressible as the x-derivative of a $\mathcal{P}$-condition of weight 4, and correspond to the integrability condition on a linear differential system which involves similar operators of order 2 and 3

[8] This is also easy to verify by direct substitution.

[9] The invariance property (91) follows again from the fact that system (76, 77) is a linearized version of the pair of $\mathcal{Y}$-constraints (74, 75) into which the invariance condition on $E_{\mathrm{Bq}}(q)$ can be decoupled.

(with an interchange of the role played by these operators). Both equations are also connected by a logarithmically linearizable transformation (Miura) to a modified NLPDE with an obvious symmetry. As to the spectral formulations (Lax systems) for KdV and PBq, they correspond both to a pair of nonlinear constraints which are linear with respect to a basis of $\mathcal{Y}$-polynomials, and compatible under a condition which is linear with respect to a basis of $\mathcal{P}$-polynomials.

8 Bilinearizability of $\mathcal{Y}$-Constraints and $\mathcal{P}$-Conditions

$\mathcal{Y}$-constraints and $\mathcal{P}$-conditions are characterized by a partitional balance between their linear and their nonlinear parts. They can therefore be "bilinearized". This follows from the one to one correspondence [14] between $\mathcal{Y}$-polynomials ($\mathcal{P}$-polynomials) and bilinear (quadratic) expressions of Hirota type:

$$\mathcal{Y}_{mx,nt}\left(v = \ln\frac{\tau'}{\tau}, w = \ln\tau'\tau\right) \equiv (\tau'\tau)^{-1} D_x^m D_t^n \tau'\cdot\tau, \tag{92}$$

$$\mathcal{P}_{mx,nt}(q = 2\ln\tau) \equiv \tau^{-2} D_x^m D_t^n \tau\cdot\tau, \quad m+n = \text{even}, \tag{93}$$

the D-operators being defined by their action on an ordered pair of tau-functions:

$$D_x^r D_t^s \tau\cdot\tau' = (\partial_x - \partial_{x'})^r (\partial_t - \partial_{t'})^s \tau(x,t)\tau'(x',t')|_{x'=x,t'=t}. \tag{94}$$

An important property of "quadratic" Hirota equations (i.e. equations in the form of a linear combination of quadratic expressions $D_x^m D_t^n \tau\cdot\tau$, $m+n =$ even set equal to zero) is the fact that they admit solutions of a generic "two soliton" type:

$$\tau_2 = 1 + e^{\theta_1} + e^{\theta_2} + A_{12}e^{\theta_1+\theta_2}, \quad \theta_i = k_i x + \omega_i t + \eta_i, \tag{95}$$

with parameters k_i and ω_i subject to a dispersion relation, and with a specific two soliton coupling factor $A_{12} = A_{12}(k_i, \omega_i)$; $\eta_i =$ arbitrary constant.

Hence, the mere observation that KdV and Bq_α can be derived from a primary $\mathcal{P}$-condition, accounts for the fact that these NLPDE's admit exact two soliton solutions of the form $u_2 = 2\partial_x^2 \ln\tau_2$.

We saw in Sect. 2 that the BH hierarchy contains NLPDE's which are linear with respect to a basis of $1+1$ dimensional Y-polynomials, and which can therefore be linearized by the Hopf-Cole transformation $u = 2\partial_x \ln\phi$. In the same way, we may now consider families of NLPDE's for $u = q_{2x}$, which are produced by differentiation of $\mathcal{P}$-conditions and which can therefore be "bilinearized" by the transformation $u = 2\partial_x^2 \ln\tau$.

However, it should be emphasized that the fact that a NLPDE can be obtained by differentiation of a $\mathcal{P}$-condition is by no means sufficient to guarantee the existence of exact N-soliton solutions with $N > 2$. Nor does it guarantee the existence of a spectral formulation of that NLPDE.

Though $\mathcal{P}$-conditions of the form:

$$E_{2n}(q) \equiv \mathcal{P}_{x,t}(q) - \mathcal{P}_{2nx}(q) = 0, \quad n = 3, 4, \ldots \tag{96}$$

may be considered in an attempt to construct higher order alternatives to KdV, it turns out that they do not produce genuine soliton equations at $n > 3$.

At $n=3$ (96) takes the form

$$E_{\mathrm{SK}}(q) \equiv \mathcal{P}_{x,t}(q) - \mathcal{P}_{6x}(q) = 0, \tag{97}$$

which, after differentiation with respect to x, yields a fifth order soliton equation for $u = q_{2x}$, known as the Sawada-Kotera equation [31]:

$$\mathrm{SK}(u) \equiv u_t - u_{5x} - 15(uu_{3x} + u_x u_{2x} + 3u^2 u_x) = 0. \tag{98}$$

A spectral formulation for SK can be derived from the primary expression $E_{\mathrm{SK}}(q)$ by application of the direct procedure of Sect. 4.4. The resulting "Lax system" is found to contain an eigenvalue problem of order 3 (in this case the auxiliary $\mathcal{Y}$-constraint of lowest possible weight which reduces $E_{\mathrm{SK}}(w+v) - E_{\mathrm{SK}}(w-v)$ to the x-derivative of a linear combination of $\mathcal{Y}$-polynomials is a constraint of weight 3). This Lax system will be obtained explicitly in Sect. 10. The fact that SK and KdV are associated with different eigenvalue equations is an indication that SK cannot be regarded as a higher order member of the KdV family (an actual 5th order member of this family will be found in Sect. 13).

9 KdV and Bq as Reductions of KP

It follows from the above considerations that 1 + 1 dimensional $\mathcal{P}$- and $\mathcal{Y}$-polynomials are not the adequate tool for the construction of higher order members of the KdV and Bq families. An extension of the above $\mathcal{P}$- and $\mathcal{Y}$-bases seems to be necessary. The way in which this extension must be realized will become clear as the comparison of the KdV- and Bq-cases is taken one step further.

9.1 Complementarity of KdV and Bq

Let us focus our attention on the "Lax pairs" associated with these two equations. To facilitate a comparison we re-label the two t-variables according to their dimension (keeping x as the basic independent variable of dimension 1), and re-scale the t-variable of KdV with a factor 1/4.

Thus, setting $t = t_2$ for Bq and $t = \frac{1}{4}t_3$ for KdV, the two $\mathcal{Y}$-systems which give rise to the corresponding Lax systems, are as follows:

KdV:

$$\mathcal{Y}_{2x}(v,w) = \lambda \qquad \text{(weight 2)}, \tag{99}$$

$$4\mathcal{Y}_{t_3}(v) - \mathcal{Y}_{3x}(v,w) - 3\lambda \mathcal{Y}_x(v) = 0 \qquad \text{(weight 3)}, \tag{100}$$

$$\text{compatibility condition:} \quad \partial_x[\mathcal{P}_{4x}(q) - 4\mathcal{P}_{x,t_3}(q)] = 0, \tag{101}$$

Bq:

$$\mathcal{Y}_{t_2}(v) - \mathcal{Y}_{2x}(v,w) = 0 \qquad \text{(weight 2)}, \tag{102}$$

$$\mathcal{Y}_{3x}(v,w) + 3\mathcal{Y}_{x,t_2}(v,w) = \mu \qquad \text{(weight 3)}, \tag{103}$$

$$\text{compatibility condition:} \quad \partial_x[\mathcal{P}_{4x}(q) + 3\mathcal{P}_{2t_2}(q)] = 0. \tag{104}$$

Comparing (99) and (102), we notice that the latter is a homogeneous $\mathcal{Y}$-constraint which contains the full set of $\mathcal{Y}$-polynomials of weight 2 that may be defined with respect to the 3 independent variables $\{x, t_2, t_3\}$, and that the former is a reduction of this constraint, obtainable by the introduction of the following additional $\mathcal{Y}$-constraint of weight 2:

$$\mathcal{Y}_{t_2}(v) = \lambda. \tag{105}$$

We also notice that (100) and (103) can be regarded as reductions of the following homogeneous $\mathcal{Y}$-constraint, which contains the full set of $\mathcal{Y}$-polynomials of weight 3 that may be defined with respect to the same independent variables:

$$4\mathcal{Y}_{t_3}(v) - \mathcal{Y}_{3x}(v, w) - 3\mathcal{Y}_{x,t_2}(v, w) = 0. \tag{106}$$

Equations (100) and (103) are obtained, respectively, by the introduction of the following additional $\mathcal{Y}$-constraints (of weight 3):

$$\mathcal{Y}_{x,t_2}(v, w) = \lambda \mathcal{Y}_x(v), \tag{107}$$

$$4\mathcal{Y}_{t_3}(v) = \mu. \tag{108}$$

We conclude that the two $\mathcal{Y}$-systems (99, 100) and (102, 103) may be regarded as complementary dimensional reductions of the following homogeneous system in $1 + 2$ dimensions:

$$\mathcal{Y}_{t_2}(v) - \mathcal{Y}_{2x}(v, w) = 0 \qquad \text{(weight 2)}, \tag{109}$$

$$4\mathcal{Y}_{t_3}(v) - \mathcal{Y}_{3x}(v, w) - 3\mathcal{Y}_{x,t_2}(v, w) = 0 \quad \text{(weight 3)}. \tag{110}$$

The compatibility of these homogeneous $\mathcal{Y}$-constraints is subject to that of the corresponding linear evolution equations for $\phi = \exp v$ (obtained after elimination of t_2-derivatives of ϕ from the linear equivalent equation (110)):

$$\phi_{t_2} = L_2(q)\phi, \qquad L_2(q) = \partial_x^2 + q_{2x}, \tag{111}$$

$$\phi_{t_3} = L_3(q_{2x}, q_{x,t_2})\phi, \qquad L_3(q_{2x}, q_{x,t_2}) = 4\partial_x^3 + 6q_{2x}\partial_x + 3(q_{3x} + q_{x,t_2}), \tag{112}$$

i.e. to a condition on $q = w - v$, which happens to be expressible as the x-derivative of a $1 + 2$ dimensional $\mathcal{P}$-condition:

$$\partial_x[\mathcal{P}_{4x}(q) + 3\mathcal{P}_{2t_2}(q) - 4\mathcal{P}_{x,t_3}(q)] = 0. \tag{113}$$

We now see that the complementarity of KdV and Bq becomes transparent when the dimensions of the independent variables are taken into account. We also notice the relevance of replacing the limited notion of “order” (with respect to x or t) by the more general (multidimensional) concept of “weight”.

9.2 The Kadomtsev-Petviashvili Equation

A remarkable point about (113) is the fact that it corresponds, after re-labelling of t_2 as a spatial variable y, to a potential version of the $2 + 1$ dimensional alternative to KdV, introduced by Kadomtsev and Petviashvili in their study of shallow water waves of small amplitude [23]:

$$\mathrm{KP}(u) \equiv (4u_{t_3} - u_{3x} - 6uu_x)_x - 3u_{2y} = 0. \tag{114}$$

The homogeneous $\mathcal{Y}$-system (109, 110) is therefore found to produce a soliton equation in two spatial dimensions, together with its associated pair of linear evolution equations (known as the "Zakharov-Shabat" formulation of KP [37]). It also produces KdV and PBq, together with their spectral formulations, by reduction: (101) and (104) are retrieved, respectively, by suppressing q's dependence with respect to t_2 (t_2-reduction) or t_3 (t_3-reduction).

9.3 Multidimensional $\mathcal{P}$- and $\mathcal{Y}$-Bases

A first element, hinting at the fundamental nature of the weight 4 condition:

$$\mathrm{KP}_4(q) \equiv 4\mathcal{P}_{x,t_3}(q) - \mathcal{P}_{4x}(q) - 3\mathcal{P}_{2t_2}(q) = 0, \tag{115}$$

is the fact that it involves the full set of $\mathcal{P}$-polynomials of weight 4, that may be defined with respect to $\{x, t_2, t_3\}$. This remains true if the set of independent variables is extended so as to include t-variables of a higher integer dimension, such as the t-variable of the SK equation (98) (which should be denoted as t_5).

It means that KP_4 is a $\mathcal{P}$-condition, with coefficients adding up to zero, which may be associated with the "sector" of weight 4 of a multidimensional "$\mathcal{P}$-basis" which comprises all $\mathcal{P}$-polynomials that may be defined with the extended set of variables $\{t_1 = x, t_2, t_3, \ldots, t_p, \ldots\}$,

$$\mathcal{P}_{n_1 t_1,\ldots,n_r t_r}(q) = e^{-q(t_i)} \partial_{t_1}^{n_1} \ldots \partial_{t_r}^{n_r} e^{q(t_i)}|_{q_{s_1 t_1,\ldots,s_r t_r}=0 \text{ if } \sum_j s_j=\text{odd}}. \tag{116}$$

This sector is constituted by the 3 $\mathcal{P}$-polynomials $\mathcal{P}_{4x}, \mathcal{P}_{x,t_3}$ and $\mathcal{P}_{2t_2}$, which correspond to the 3 partitions of 4 into an even number of parts.

A stronger hint at the fundamental nature of KP_4 comes from the observation that it arises as a potential version of the compatibility condition on two homogeneous $\mathcal{Y}$-constraints:

$$\mathrm{mKP}_2(v, w) \equiv \mathcal{Y}_{t_2}(v) - \mathcal{Y}_{2x}(v, w) = 0, \tag{117}$$

$$\mathrm{mKP}_3(v, w) \equiv 4\mathcal{Y}_{t_3}(v) - \mathcal{Y}_{3x}(v, w) - 3\mathcal{Y}_{x,t_2}(v, w) = 0, \tag{118}$$

which stand out as a fundamental pair (of weight 2 and 3) with respect to a multidimensional "$\mathcal{Y}$-basis", constituted by the polynomials ($t_1 = x$):

$$\mathcal{Y}_{n_1 t_1,\ldots,n_r t_r}(v, w) = e^{-v(t_i)} \partial_{t_1}^{n_1} \ldots \partial_{t_r}^{n_r} e^{v(t_i)}|_{v_{s_1 t_1,\ldots,s_r t_r}=w_{s_1 t_1,\ldots,s_r t_r} \text{ if } \sum_j s_j=\text{even}}. \tag{119}$$

Indeed, mKP_2 is the only homogeneous $\mathcal{Y}$-constraint, with coefficients adding up to zero, which contains the full set of polynomials that belong to the $\mathcal{Y}$-sector of weight 2, whereas mKP_3 is the only such constraint of weight 3 which happens to be compatible with mKP_2 under a single nonlinear condition on $q = w - v$.

The first assertion is obvious; the second one is easy to verify at the level of the linear evolution equations which correspond to mKP_2 and the most general homogeneous $\mathcal{Y}$-constraint of weight 3, with coefficients adding up to zero ($\alpha =$ parameter):

$$\mathcal{Y}_{t_3}(v) - \alpha\mathcal{Y}_{3x}(v, w) - (1 - \alpha)\mathcal{Y}_{x,t_2}(v, w) = 0. \tag{120}$$

The integrability of that system is found to be subject to a single condition on q iff $\alpha = \frac{1}{4}$.

We end this section by noticing that mKP_2 and mKP_3 can also be recovered from KP_4, by looking for the simplest homogeneous $\mathcal{Y}$-constraint (with coefficients adding up to zero) which reduces the invariance condition

$$\mathrm{KP}_4(w + v) - \mathrm{KP}_4(w - v) \equiv 2(4v_{x,t_3} - v_{4x} - 3v_{2t_2} - 6v_{2x}w_{2x}) = 0, \tag{121}$$

to the x-derivative of another $\mathcal{Y}$-constraint (the former being mKP_2, the latter mKP_3).

9.4 A Modified KP Equation

There is reason to expect that the fundamental pair (mKP_2, mKP_3) produces a modified version of KP, in addition to KP itself.

This is easily confirmed by rewriting (117) and (118) as a pair of constraints on v and $q = w - v$, and by differentiating the latter with respect to x. Elimination of q_{2x,t_2} from this last equation yields a $1 + 2$ dimensional NLPDE for v, which, after a re-labelling of t_2 as y, corresponds to a modified version of KP:

$$\mathrm{mKP}(v) \equiv (4v_{t_3} - v_{3x} + 2v_x^3)_x - 6v_{2x}v_y - 3v_{2y} = 0. \tag{122}$$

Equation (122) displays an obvious symmetry under the transformation ($v \to -v$, $y \to -y$). Its solutions $\{v(x, y, t_3), -v(x, -y, t_3)\}$ produce pairs of solutions $\{u, \tilde{u}\}$ to the KP equation, according to the generalized Miura link (given by mKP_2, with $w_{2x} = v_{2x} + u$ and $t_2 = y$):

$$u = v_y - v_{2x} - v_x^2, \tag{123}$$

$$\tilde{u} = v_y + v_{2x} - v_x^2. \tag{124}$$

Again on finds that:

$$\tilde{u} = u + 2v_{2x}. \tag{125}$$

10 The BKP_6 Equation

We have seen that KdV and Bq can be obtained, together with their spectral formulations, from the fundamental set $\{\mathrm{mKP}_2, \mathrm{mKP}_3, \mathrm{KP}_4\}$, by means of appropriate dimensional reductions.[10]

Let us now see if there exists a similar set of fundamental equations (of higher weight) from which the SK equation (98) and its spectral formulation may be derived.

If the primary SK equation (97) is to be obtained by reduction of an alternative to KP_4, this alternative should take the form of a homogeneous $1 + 2$ dimensional $\mathcal{P}$-condition of weight 6. The x-derivative of this condition should coincide with the compatibility condition on a pair of uniquely determined $\mathcal{Y}$-constraints, comparable to mKP_2 and mKP_3. The one of highest weight (weight 5) should involve 3 variables (x, t_5 and t_r), the one of lowest weight being a $1 + 1$ dimensional constraint of weight r (involving x and t_r).

10.1 The Fundamental Pair (DKP_3, DKP_5)

The simplest alternative to mKP_2 that may be considered, is a homogeneous $1 + 1$ dimensional $\mathcal{Y}$-constraint of weight 3, with coefficients adding up to zero. This $\mathcal{Y}$-constraint is uniquely determined if t_2 is excluded from the set of independent variables. It then takes the form:

$$\mathrm{DKP}_3(v, w) \equiv \mathcal{Y}_{t_3}(v) - \mathcal{Y}_{3x}(v, w) = 0. \tag{126}$$

[10] The Lax system for KdV (Bq) is a linear equivalent of a $t_2(t_3)$-reduction of (mKP_2, mKP_3). The t_2-reduction is obtained by setting $v_{t_2} = \lambda$ and $q_{x,t_2} = 0$ (on account of the additional constraints (105, 107)); the t_3-reduction is obtained bt setting $4v_{t_3} = \mu$ (on account of the additional constraint (108)). The Lax systems can also be obtained directly from the linear system (111, 112) by setting $\phi_{t_2} = \lambda\phi$ and $q_{x,t_2} = 0$ for KdV, and $4\phi_{t_3} = \mu\phi$ for Bq.

Furthermore, if both t_2 and t_4 are excluded, there is only one homogeneous $\mathcal{Y}$-constraint of weight 5 (with coefficients adding up to zero) which involves the full set of $\mathcal{Y}$-polynomials of that weight, and which happens to be compatible with DKP_3 under a single nonlinear condition on $q = w - v$. It is the constraint:

$$\mathrm{DKP}_5(v, w) \equiv \mathcal{Y}_{t_5}(v) - \frac{1}{6}\mathcal{Y}_{5x}(v, w) - \frac{5}{6}\mathcal{Y}_{2x,t_3}(v, w) = 0. \tag{127}$$

The nonlinear compatibility condition on the pair (DKP_3, DKP_5) is the integrability condition for the corresponding pair of linear evolution equations for $\phi = \exp v$:

$$\phi_{t_3} = \phi_{3x} + 3q_{2x}\phi_x, \tag{128}$$

$$\phi_{t_5} = \phi_{5x} + 5q_{2x}\phi_{3x} + 5q_{3x}\phi_{2x} + \left(\frac{10}{3}q_{4x} + \frac{10}{3}q_{x,t_3} + 5q_{2x}^2\right)\phi_x. \tag{129}$$

This condition is a homogeneous NLPDE for q (of weight 7) which happens to be expressible as the x-derivative of the following $\mathcal{P}$-condition of weight 6:

$$\mathrm{BKP}_6(q) \equiv \mathcal{P}_{6x}(q) + 9\mathcal{P}_{x,t_5}(q) - 5\mathcal{P}_{3x,t_3}(q) - 5\mathcal{P}_{2t_3}(q) = 0. \tag{130}$$

The set $\{\mathrm{DKP}_3, \mathrm{DKP}_5, \mathrm{BKP}_6\}$ displays a striking analogy with the former set $\{\mathrm{mKP}_2, \mathrm{mKP}_3, \mathrm{KP}_4\}$. The pairs ($\mathrm{mKP}_2$, mKP_3) and (DKP_3, DKP_5) may be regarded as fundamental pairs of $\mathcal{Y}$-constraints, the former with respect to the "complete" multidimensional $\mathcal{Y}$-basis, the latter with respect to a "restricted" $\mathcal{Y}$-basis, from which all elements involving *even* dimensional variables t_{2n} have been eliminated.

As in the previous case one may verify that DKP_3 and DKP_5 are obtainable from BKP_6, as a pair of homogeneous $\mathcal{Y}$-constraints into which the invariance condition

$$\mathrm{BKP}_6(w + v) - \mathrm{BKP}_6(w - v) = 0 \tag{131}$$

can be decoupled (DKP_3 is the simplest auxiliary $\mathcal{Y}$-constraint which reduces this condition to the x-derivative of another $\mathcal{Y}$-constraint, the latter being DKP_5).

10.2 Primary SK and Ramani Equations as Reductions of BKP_6

It is clear that the primary SK equation (97) is a re-scaled version of the t_3-reduction of BKP_6 (obtained by setting $q_{t_3} = 0$ and $t_5 = -9t$).

As to the spectral formulation of SK, it can either be determined as a (re-scaled) linear counterpart to the t_3-reduction of (DKP_3, DKP_5), obtained by imposing the additional $\mathcal{Y}$-constraints:

$$\mathcal{Y}_{t_3}(v) = \lambda, \qquad \mathcal{Y}_{2x,t_3}(v, w) = \lambda\mathcal{Y}_{2x}(v, w), \tag{132}$$

or derived directly from the linear system (128, 129) by setting: $\phi_{t_3} = \lambda\phi$, $q_{x,t_3} = 0$ and $t_5 = -9t$. It can therefore be expressed as follows ($u = q_{2x}$):

$$\phi_{3x} + 3u\phi_x = \lambda\phi, \tag{133}$$

$$\phi_t + 9\phi_{5x} + 45u\phi_{3x} + 45u_x\phi_{2x} + (30u_{2x} + 45u^2)\phi_x = 0. \tag{134}$$

There is a striking similarity between the ways in which KdV and SK arise, respectively, as a t_2-reduction of KP_4 and a t_3-reduction of BKP_6. This suggests the existence of a complementary $1 + 1$ dimensional soliton equation, that would arise as a t_5-reduction of BKP_6 (as a counterpart to Bq which arose as a t_3-reduction of KP_4): it is the Ramani equation [30], a primary version of which is the $\mathcal{P}$-condition:

$$E_{\mathrm{Ram}}(q) \equiv \mathcal{P}_{6x}(q) - 5\mathcal{P}_{3x,t_3}(q) - \mathcal{P}_{2t_3}(q) = 0. \tag{135}$$

A spectral formulation of the Ramani equation can be derived straight away as a t_5-reduction of the linear system (133, 134), obtained by setting $\phi_{t_5} = \mu\phi$.

11 Higher Weight Members of the KP Family

The $\mathcal{P}$-conditions KP_4 and BKP_6, generated by the fundamental pairs (mKP_2, mKP_3) and (DKP_3, DKP_5), are expected to represent *basic* members of two families of $\mathcal{P}$-conditions (one with respect to the complete $\mathcal{P}$-basis, the other one with respect to a restricted $\mathcal{P}$-basis, from which polynomials involving *even* dimensional variables have been eliminated), the higher weight members of which could be generated similarly by pairs of "conditionally" compatible $\mathcal{Y}$-constraints of a higher weight.

11.1 The mKP_4 Family

To elaborate this idea one may try to identify $\mathcal{P}$-conditions corresponding to higher KP-members.

Let us therefore consider the "$\mathcal{Y}$-sector" of weight 4, and the corresponding parameter family of homogeneous $\mathcal{Y}$-constraints:

$$\mathcal{Y}_{t_4} = \alpha_1\mathcal{Y}_{4x} + \alpha_2\mathcal{Y}_{x,t_3} + \alpha_3\mathcal{Y}_{2x,t_2} + (1-\alpha_1-\alpha_2-\alpha_3)\mathcal{Y}_{2t_2}. \tag{136}$$

Members of this family, which are compatible with mKP_2 and mKP_3 under KP_4 and further nonlinear conditions on q (of a higher weight), may be identified by considering the linear evolution equations for $\phi = \exp v$ which correspond to mKP_2, mKP_3 and the constraints (136):

$$\phi_{t_2} = \phi_{2x} + q_{2x}\phi, \tag{137}$$

$$\phi_{t_3} = \phi_{3x} + \frac{3}{2}q_{2x}\phi_x + \frac{3}{4}(q_{3x} + q_{x,t_2})\phi, \tag{138}$$

$$\begin{aligned}\phi_{t_4} = \phi_{4x} &+ \left(2 + 4\alpha_1 - \frac{1}{2}\alpha_2\right)q_{2x}\phi_{2x} \\ &+ \left[\left(2 - 2\alpha_1 + \frac{1}{4}\alpha_2\right)q_{3x} + \left(\frac{3}{4}\alpha_2 + \alpha_3\right)q_{x,t_2}\right]\phi_x \\ &+ \left[\left(1 - \frac{1}{4}\alpha_2\right)q_{4x} + \alpha_2 q_{x,t_3} + \left(1 - \alpha_1 - \frac{1}{4}\alpha_2 - \alpha_3\right)q_{2x,t_2} + (1-\alpha_1-\alpha_2-\alpha_3)q_{2t_2}\right. \\ &\left.+ (1 + 2\alpha_1 - \alpha_2)q_{2x}^2\right]\phi.\end{aligned} \tag{139}$$

Inspection of (137) and (139) shows that their compatibility is subject to a set of nonlinear conditions on q iff:

$$\alpha_2 = 8\alpha_1, \qquad \alpha_3 = \frac{1}{2} - 3\alpha_1. \tag{140}$$

These restrictions reduce the above family (136) to the one parameter family:

$$\mathrm{mKP}_4(\alpha_1) \equiv \mathcal{Y}_{t_4} - \frac{1}{2}\mathcal{Y}_{2x,t_2} - \frac{1}{2}\mathcal{Y}_{2t_2} - \alpha_1(\mathcal{Y}_{4x} + 8\mathcal{Y}_{x,t_3} - 3\mathcal{Y}_{2x,t_2} - 6\mathcal{Y}_{2t_2}) = 0, \quad (141)$$

each member of which corresponds, under condition KP_4, to the same evolution equation:

$$\phi_{t_4} = L_4(q)\phi, \quad (142)$$

$$L_4(q) = \partial_x^4 + 2q_{2x}\partial_x^2 + (2q_{3x} + q_{x,t_2})\partial_x + \left(q_{4x} + \frac{1}{2}q_{2x,t_2} + \frac{1}{2}q_{2t_2} + \frac{1}{2}q_{2x}^2\right). \quad (143)$$

11.2 The Pair (KP_4, KP_5)

The compatibility of (137) and (142) is now found to be subject to nonlinear conditions on q which, under KP_4, are satisfied if q satisfies a single new condition (of weight 6) which, on account of KP_4, happens to be expressible as the x-derivative of the following $\mathcal{P}$-condition of weight 5:

$$\mathrm{KP}_5(q) \equiv 3\mathcal{P}_{x,t_4}(q) - 2\mathcal{P}_{t_2,t_3}(q) - \mathcal{P}_{3x,t_2}(q) = 0. \quad (144)$$

Furthermore, it is a straightforward matter to check that the compatibility of (138) and (142) is subject to 3 nonlinear conditions on q (of weight 5, 6 and 7), which are all satisfied if q satisfies KP_4 and KP_5.

Hence, we conclude that $\{\mathrm{mKP}_2, \mathrm{mKP}_3$ and $\mathrm{mKP}_4(\alpha_1)\}$ constitute a set of $\mathcal{Y}$-constraints (of weight 2, 3 and 4) which, for any value of the parameter α_1, are mutually compatible under the pair $(\mathrm{KP}_4, \mathrm{KP}_5)$.

11.3 $\mathrm{mKP}_4(t_{i\le 3})$ and $\mathrm{mKP}_4(t_{i\le 4})$

The mKP_4 (141) form a linear 2-dimensional sub-space of $\mathcal{Y}$-constraints of weight 4, with basis "vectors":

$$\mathrm{mKP}_4(t_{i\le 3}) \equiv \mathcal{Y}_{4x} - 3\mathcal{Y}_{2x,t_2} - 6\mathcal{Y}_{2t_2} + 8\mathcal{Y}_{x,t_3} = 0, \quad (145)$$

$$\mathrm{mKP}_4(t_{i\le 4}) \equiv \mathcal{Y}_{t_4} - \frac{1}{2}\mathcal{Y}_{2x,t_2} - \frac{1}{2}\mathcal{Y}_{2t_2} = 0. \quad (146)$$

Constraint (145) involves only variables of dimension up to 3: it is a differential consequence of the fundamental constraints mKP_2 and mKP_3 (under the assumption of their compatibility). Constraint (146) involves t_4, and may therefore be considered as a *basic* weight 4 member of the mKP family.

Higher weight members of the primary KP hierarchy (of which KP_4 and KP_5 appear to be members of weight 4 and 5), but which do *not* involve other variables than x, t_2, t_3 and t_4—they correspond to differential consequences of KP_4 and KP_5—can be obtained from compatibility conditions of weight 7 and 8 that are related to the above linear evolution equations (137, 138, 142). These members can be found in [25], where they have been denoted as $\mathrm{KP}_6(t_{i\le 4})$, $\mathrm{KP}_7(t_{i\le 4})$ and $\mathrm{KP}_8(t_{i\le 4})$.

12 A Higher Order Member of the PBq Family

Inspection of (144) shows that a t_r-reduction of KP_5 alone (with $r = 2$ or $r = 3$), does *not* produce any $\mathcal{P}$-condition that might provide us with the primary version of a $1 + 1$ dimensional soliton equation.

Yet, keeping in mind that KP_5 must be considered in association with KP_4, one ought to examine reductions of the pair (KP_4, KP_5).

12.1 t_3-Reduction of (KP_4, KP_5)

A t_3-reduction of (115) and (144) produces the system:

$$\mathcal{P}_{4x}(q) + 3\mathcal{P}_{2t_2}(q) = 0, \tag{147}$$

$$3\mathcal{P}_{x,t_4}(q) - \mathcal{P}_{3x,t_2}(q) = 0. \tag{148}$$

Equation (147) is the primary version of the Bq equation (9), or of the corresponding coupled system of evolution equations (79, 80) for the variables u and r (in which $t = t_2$).

On the other hand it is clear that differentiation of (148), with respect to x and t_2, produces the $1 + 2$ dimensional system:

$$u_{t_4} = \frac{1}{3} r_{3x} + (ur)_x, \tag{149}$$

$$r_{t_4} = \frac{1}{3} r_{2x,t_2} + (ur)_{t_2}, \tag{150}$$

which, by means of (79, 80) with $t = t_2$, can be reduced to the $1 + 1$ dimensional system:

$$u_{t_4} = \frac{1}{3} r_{3x} + (ur)_x, \tag{151}$$

$$r_{t_4} = -\frac{1}{9} u_{5x} - uu_{3x} - 2u_x u_{2x} - 2u^2 u_x + rr_x. \tag{152}$$

System (151, 152) should, by construction, be closely linked to the PBq system.

The link is easily clarified by looking for a spectral formulation of system (151, 152) and by comparing it with the "Lax system" (76, 77) for PBq.

Application of the direct procedure of Sect. 4.4 to the system (147, 148) leads to a pair of invariance conditions:

$$E_{\mathrm{Bq}}(w + v) - E_{\mathrm{Bq}}(w - v) = 0, \tag{153}$$

$$\mathrm{KP}_5(w + v)|_{t_3\text{-red}} - \mathrm{KP}_5(w - v)|_{t_3\text{-red}} = 0 \tag{154}$$

which can be decoupled into the 3 $\mathcal{Y}$-constraints that correspond to the t_3 reductions of the constraints mKP_2, mKP_3 and $mKP_4(t_{i\le 4})$. These reductions are: mKP_2, $mKP_4(t_{i\le 4})$ and (103).

These constraints give rise to a linear system for $\phi = \exp v$ which comprises the *eigenvalue* equation associated with Bq, together with a linear evolution equation with respect to t_4, which, under condition (147), can be reduced to (142). The compatibility of these linear equations is subject to the conditions (151, 152), as indicated by the commutator relation:

$$[\partial_{t_4} - L_4, L_3] = \frac{3}{2}\left[u_{t_4} - \frac{1}{3}r_{3x} - (ur)_x\right]\partial_x + \frac{3}{4}\left\{\left[u_{t_4} - \frac{1}{3}r_{3x} - (ur)_x\right]_x + \left(r_{t_4} + \frac{1}{9}u_{5x} + uu_{3x} + 2u_x u_{2x} + 2u^2 u_x - rr_x\right)\right\}. \tag{155}$$

The analogy between the commutator relations (83) and (155) provides sufficient grounds for regarding the systems (79, 80) and (151, 152) as members of a same PBq family. Only the basic PBq member can be represented by a $1+1$ dimensional $\mathcal{P}$-condition. The next one admits a $1+2$ dimensional "$\mathcal{P}$-representation", consisting of the primary version of the former (with the "Bq time" as an auxiliary variable t_2), and another $\mathcal{P}$-condition involving the 3 variables x, t_2 and t_4.

13 A Higher Member of the KdV Family

A t_2-reduction of the pair (KP_4, KP_5) does not produce anything else than KdV. However, a higher member of the KdV family might be obtained from a $1+2$ dimensional $\mathcal{P}$-system, consisting of a t_2-reduction of KP_4 and a $\mathcal{P}$-condition that would correspond to a t_2-reduction of a weight 6 member of the primary KP family (that has not been identified yet).

13.1 The Set (KP_4, KP_5, KP_6)

In order to find out, we take the analysis of Sect. 11.1 one step further. We consider the $\mathcal{Y}$-sector of weight 5, and the corresponding 5 parameter family of homogeneous $\mathcal{Y}$-constraints (with coefficients adding up to zero):

$$\mathcal{Y}_{t_5} = \alpha_1\mathcal{Y}_{5x} + \alpha_2\mathcal{Y}_{3x,t_2} + \alpha_3\mathcal{Y}_{2x,t_3} + \alpha_4\mathcal{Y}_{x,t_4} + \alpha_5\mathcal{Y}_{x,2t_2} + \left(1 - \sum_{i=1}^{5}\alpha_i\right)\mathcal{Y}_{t_2,t_3}. \tag{156}$$

As this equation is taken in association with mKP_2, mKP_3 and $mKP_4(t_{i\le 4})$, it can be transformed into a linear evolution equation for $\phi = \exp v$ with respect to t_5. Compatibility of this evolution equation with the linear counterpart to mKP_2:

$$\phi_{t_2} = \phi_{2x} + q_{2x}\phi \tag{157}$$

is subject to 3 nontrivial conditions on q (of weight 5, 6 and 7) iff:

$$\alpha_3 = \frac{1}{4} - 4\alpha_1, \qquad \alpha_4 = 18\alpha_1 + 6\alpha_2, \qquad \alpha_5 = \frac{1}{2} - 3\alpha_1 - 3\alpha_2. \tag{158}$$

Under these restrictions, one obtains a linear evolution equation with respect to t_5 of the form:

$$\phi_{t_5} = L_5(q; \alpha_1, \alpha_2),$$

$$\begin{aligned}L_5(q;\alpha_1,\alpha_2) &= \partial_x^5 + \frac{5}{2}q_{2x}\partial_x^3 + \left[\frac{15}{4}q_{3x} + \frac{5}{4}q_{x,t_2}\right]\partial_x^2 \\ &+ \left[\frac{5}{4}q_{2x,t_2} - \frac{35}{4}q_{2t_2} + \frac{25}{2}q_{x,t_3} - \frac{15}{2}q_{2x}^2 - 2\alpha_1 \mathrm{KP}_4\right]\partial_x \\ &+ \left\{\frac{15}{16}q_{5x} + \frac{5}{16}q_{x,2t_2} + \frac{5}{6}q_{3x,t_2} + \frac{5}{12}q_{t_2,t_3} + \frac{15}{8}q_{2x}q_{3x} + \frac{5}{4}q_{2x}q_{x,t_2}\right. \\ &\left. - (3\alpha_1+\alpha_2)[(\mathrm{KP}_4)_x - 2\mathrm{KP}_5]\right\}. \end{aligned} \tag{159}$$

This equation turns out to be compatible with (157), under the previous conditions KP_4 and KP_5, if q satisfies an additional condition of weight 7, which happens to be expressible as the x-derivative of the following $\mathcal{P}$-condition of weight 6 (its coefficients still add up to zero):

$$\mathrm{KP}_6(t_{i\le 5}) \equiv 144\mathcal{P}_{x,t_5} + \mathcal{P}_{6x} - 45\mathcal{P}_{2x,2t_2} - 20\mathcal{P}_{3x,t_3} - 80\mathcal{P}_{2t_3} = 0. \tag{160}$$

13.2 The Lax Equation

Let us now consider the t_2-reduction of the pair (KP_4, KP_6):

$$4\mathcal{P}_{x,t_3}(q) - \mathcal{P}_{4x}(q) = 0, \tag{161}$$

$$144\mathcal{P}_{x,t_5}(q) + \mathcal{P}_{6x}(q) - 20\mathcal{P}_{3x,t_3}(q) - 80\mathcal{P}_{2t_3}(q) = 0. \tag{162}$$

Differentiation of (162) with respect to x produces, after elimination of all terms with t_3 derivatives by means of (161), a $1+1$ dimensional NLPDE for $u = q_{2x}$, with terms similar to those of the SK equation (98) (notice that (162) can be transformed into BKP_6 by a rescaling of t_5 and t_3). This NLPDE, known as the Lax equation [26] (upon rescaling t_5 by a factor 16):

$$\mathrm{Lax}(u) \equiv u_{t_5} - u_{5x} - 10uu_{3x} - 20u_x u_{2x} - 30u^2 u_x = 0, \tag{163}$$

should, by construction, be closely related to KdV. The relation can again be clarified at the level of the corresponding spectral formulations.

Let us therefore see if a spectral formulation of (163) can be obtained through application of the direct procedure to its potential version, which (in terms of the dimensionless variable q, introduced by setting $u = q_{2x}$) takes the form:

$$E_{\mathrm{Lax}}(q) \equiv q_{x,t_5} - q_{6x} - 10q_{2x}q_{4x} - 5q_{3x}^2 - 10q_{2x}^3 = 0. \tag{164}$$

In view of the fact that (164) is the potential version of an NLPDE which admits the $\mathcal{P}$-representation (161, 162), it suffices to examine whether the invariance condition:

$$E_{\mathrm{Lax}}(w+v) - E_{\mathrm{Lax}}(w-v) = 0 \tag{165}$$

can be reduced to the x-derivative of a $\mathcal{Y}$-constraint of weight 5, by means of the auxiliary $\mathcal{Y}$-constraint of weight 2 that produced the Lax system for KdV. It is easy to verify that condition (165) can indeed be decoupled into the following pair of λ-dependent $\mathcal{Y}$-constraints:

$$\mathcal{Y}_{2x}(v,w) = \lambda, \tag{166}$$

$$\mathcal{Y}_{t_5}(v) - \mathcal{Y}_{5x}(v,w) - 15\lambda^2\mathcal{Y}_x(v) = 0. \tag{167}$$

The linear equivalent to these constraints comprises the eigenvalue equation associated with KdV, as well as a linear evolution equation for $\phi = \exp v$, which (after elimination of its λ-dependent terms) can be expressed as follows ($u = q_{2x}$):

$$\phi_{t_5} = B_5(u)\phi,$$
$$B_5(u) = 16\partial_x^5 + 40u\partial_x^3 + 60u_x\partial_x^2 + (50u_{2x} + 30u^2)\partial_x + (15u_{3x} + 30uu_x). \tag{168}$$

The integrability of this linear system is subject to a condition on u which coincides with the Lax equation (163). The latter can therefore be reformulated as an operator equation in close analogy with KdV:

$$[\partial_{t_5} - B_5, L_2] = 0. \tag{169}$$

13.3 $\mathcal{P}$-Basis Decomposition of the Potential Lax Equation

Equation (169) and the spectral formulation of (163) show clearly that the Lax equation is to be considered as a "higher" KdV equation.

Yet, it should be emphasized that the operator representation (169) was obtained from the invariance condition (165) with the *a priori* knowledge of a connection between (163) and KdV (which suggested (166) as an adequate auxiliary $\mathcal{Y}$-constraint). This raises the question as to whether the link between Lax and KdV could also have been obtained *without* this information (i.e. without the "$\mathcal{P}$-representation" (161, 162) of (163)).

We shall now see that this is the case: an *explicit* link between Lax and KdV can be disclosed by looking for a "$\mathcal{P}$-basis" decomposition of the potential version (164) of the Lax equation.

The way in which this can be done is indicated by the following elementary identities which relate $\mathcal{P}$-polynomials from "adjacent" $\mathcal{P}$-sectors:

$$(\partial_x^2 + 3q_{2x})\mathcal{P}_{x,t}(q) \equiv \mathcal{P}_{3x,t}(q), \tag{170}$$

$$(\partial_x^2 + 3q_{2x})\mathcal{P}_{4x}(q) \equiv \mathcal{P}_{6x}(q) - 6(q_{2x}q_{4x} - q_{3x}^2 + q_{2x}^3). \tag{171}$$

They are seen to allow a reduction of:

$$\begin{aligned} E_{\mathrm{Lax}}(q) &\equiv \mathcal{P}_{x,t_5}(q) - \mathcal{P}_{6x}(q) + 5(q_{2x}q_{4x} - q_{3x}^2 + q_{2x}^3) \\ &\equiv \mathcal{P}_{x,t_5}(q) - \frac{1}{6}\mathcal{P}_{6x}(q) - \frac{5}{6}(\partial_x^2 + 3q_{2x})\mathcal{P}_{4x}(q) \end{aligned} \tag{172}$$

to a linear combination of $\mathcal{P}$-polynomials of weight 6, upon the introduction of an *auxiliary* independent variable of weight 3, say t_3, together with an *auxiliary* $\mathcal{P}$-condition of the form $\mathcal{P}_{4x}(q) \div \mathcal{P}_{x,t_3}(q)$.

Thus, it is found that the primary KdV equation shows up "naturally" as the auxiliary $\mathcal{P}$-condition of weight 4 (with coefficients adding up to zero and with its auxiliary variable t_3) which allows a "$\mathcal{P}$-basis" decomposition of (164).

Let us compare of the $\mathcal{P}$-system which arises from this decomposition (the coefficients in the two conditions add up to zero):

$$\mathcal{P}_{x,t_3}(q) - \mathcal{P}_{4x}(q) = 0, \tag{173}$$

$$\mathcal{P}_{x,t_5}(q) - \frac{1}{6}\mathcal{P}_{6x}(q) - \frac{5}{6}\mathcal{P}_{3x,t_3}(q) = 0, \tag{174}$$

with the former $\mathcal{P}$-system (161, 162).

In contrast with (162), which contains the three $\mathcal{P}$-polynomials of weight 6 that may be defined with respect to x and t_3:

$$\{\mathcal{P}_{6x}(q), \mathcal{P}_{3x,t_3}(q), \mathcal{P}_{2t_3}(q)\}, \tag{175}$$

we see that (174) involves only two members of this set. The absence in this condition of a term proportional to $\mathcal{P}_{2t_3}(q)$ is an indication that (173) introduces a *linear dependence* between these members (or between their derivatives with respect to x). This is easily verified by showing that the primary KdV condition implies the relation:

$$\partial_x[\mathcal{P}_{6x}(q) + \mathcal{P}_{3x,t_3}(q) - 2\mathcal{P}_{2t_3}(q)] = 0. \tag{176}$$

It means that the systems (161, 162) and (173, 174) are two representatives of a parameter family of equivalent $1+2$ dimensional $\mathcal{P}$-representations of the Lax equation (163), all of which contain the primary KdV equation with its auxiliary variable t_3, as its basic component ($\alpha =$ parameter):

$$\mathcal{P}_{x,t_3}(q) - \mathcal{P}_{4x}(q) = 0, \tag{177}$$

$$\mathcal{P}_{x,t_5}(q) + \frac{(\alpha-1)}{6}\mathcal{P}_{6x}(q) + \frac{(\alpha-5)}{6}\mathcal{P}_{3x,t_3}(q) - \frac{\alpha}{3}\mathcal{P}_{2t_3}(q) = 0. \tag{178}$$

One can check that (177) and (178) admit[11] a common set of potential two soliton solutions:

$$\begin{aligned} &q_2 = 2\ln\tau_2, \quad \tau_2 = 1 + e^{\theta_1} + e^{\theta_2} + A_{12}^{\text{KdV}} e^{\theta_1+\theta_2}, \\ &\theta_i = k_i x + k_i^3 t_3 + k_i^5 t_5 + \eta_i, \quad A_{12}^{\text{KdV}} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}. \end{aligned} \tag{179}$$

14 $\mathcal{P}$-Representations and the Identification of Soliton Equations

We have seen that the Lax equation could be identified as a sech squared soliton system belonging to the KdV hierarchy. This equation was found to admit a family of $1+2$ dimensional $\mathcal{P}$-representations.

Nonlinear evolution equations in $1+1$ dimensions which admit such a representation, in which both $\mathcal{P}$-conditions share a common set of two soliton solutions, stand a good chance of being true soliton equations.

To test this idea we may try to identify soliton equations among the following family of shallow water wave equations [33]:

$$E_{\alpha,\beta}(V) \equiv V_{3x,t} - V_{2x} + \alpha V_{x,t} + \beta V_x V_{x,t} + V_{2x} V_t = 0, \tag{180}$$

by looking for members which admit an appropriate $\mathcal{P}$-representation (method of "$\mathcal{P}$-basis" decomposition).

[11] This follows immediately from the corresponding quadratic Hirota equations, when $\alpha \neq \frac{5}{3}$. The case $\alpha = \frac{5}{3}$ is the one for which (178) coincides with BKP_6.

Setting $V = cq_x$, $c =$ dimensionless constant, on account of the dimensions of the variables ($\dim x = 1$ implies $\dim V = \dim t = -1$ and $\dim \alpha = -2$) we find that, as $\beta \neq -1$, one may choose $c = \frac{6}{1+\beta}$ so as to express (180) as follows:

$$\left(\frac{\beta-1}{\beta+1}\right)[\mathcal{P}_{4x}(q)]_t + \left[\frac{2}{1+\beta}\mathcal{P}_{3x,t}(q) - \mathcal{P}_{2x}(q) + \alpha\mathcal{P}_{x,t}(q)\right]_x = 0. \tag{181}$$

14.1 The HS Shallow Water Wave Equation

If $\beta = 1$ the above equation is expressible as the x-derivative of a $1+1$ dimensional $\mathcal{P}$-condition:

$$E_{\text{HS}}(q) \equiv \mathcal{P}_{3x,t}(q) - \mathcal{P}_{2x}(q) + \alpha\mathcal{P}_{x,t}(q) = 0. \tag{182}$$

Equation (182) may be regarded as the primary version of an equation for $u = q_{2x} = \frac{1}{3}V_x$, which is known as the *Hirota-Satsuma* shallow water wave equation [19]:

$$\text{HS}(u) \equiv u_{2x,t} - u_x + \alpha u_t + 3uu_t + 3u_x \int_{-\infty}^{x} u_t dx' = 0. \tag{183}$$

We verify that this equation admits a spectral formulation, by applying the direct procedure of Sect. 4.4 to (182). It is easy to see that the invariance condition on $E_{\text{HS}}(q)$ can be expressed in a way suggestive of a simple $\mathcal{Y}$-basis decomposition:

$$\begin{aligned} E_{\text{HS}}(w+v) - E_{\text{HS}}(w-v) &\equiv 2(v_{2x} + \alpha v_{x,t} - v_{3x,t} \\ &\quad - 3v_{2x}w_{x,t} - 3v_{x,t}w_{2x}) \\ &\equiv 2\{[\mathcal{Y}_x(v)]_x + [\alpha\mathcal{Y}_x(v) - \mathcal{Y}_{3x}(v,w)]_t \\ &\quad + 3W[\mathcal{Y}_x(v), \mathcal{Y}_{x,t}(v,w)]\} = 0. \end{aligned} \tag{184}$$

Thus, it is found that condition (184) can be reduced to the t-derivative of a $\mathcal{Y}$-constraint of weight 3, by the introduction of an auxiliary $\mathcal{Y}$-constraint of weight zero (notice that in this case $\dim t = -\dim x$). It produces the following $\mathcal{Y}$-basis decomposition (λ is an integration constant):

$$\mathcal{Y}_{x,t}(v,w) = \frac{1}{3} \quad \text{(weight zero)}, \tag{185}$$

$$\mathcal{Y}_{3x}(v,w) - \alpha\mathcal{Y}_x(v) = \lambda \quad \text{(weight 3)}. \tag{186}$$

The integrability of the corresponding linear system for $\phi = \exp v$:

$$\phi_{x,t} + \left(q_{x,t} - \frac{1}{3}\right)\phi = 0, \tag{187}$$

$$\phi_{3x} + (3q_{2x} - \alpha)\phi_x = \lambda\phi, \tag{188}$$

is subject to a condition on q which coincides with (183) expressed in terms of q:

$$\partial_x E_{\text{HS}}(q) \equiv [\mathcal{P}_{3x,t}(q) - \mathcal{P}_{2x}(q) + \alpha\mathcal{P}_{x,t}(q)]_x = 0. \tag{189}$$

It means that system (187, 188) constitutes a spectral formulation of the HS equation.

The presence in this system of a third order eigenvalue equation which, by the substitution $u = q_{2x} - \frac{\alpha}{3}$, can be transformed into the eigenvalue equation (133) associated with SK, tells us that the "shifted" version of HS (i.e. (183) taken with $\alpha = 0$), should be regarded as a member of the SK family.

Inspection of the corresponding primary equations:

$$E_{\mathrm{HS},\alpha=0}(q) \equiv \mathcal{P}_{3x,t}(q) - \mathcal{P}_{2x}(q) = 0, \tag{190}$$

$$E_{\mathrm{SK}}(q) \equiv \mathcal{P}_{x,t}(q) - \mathcal{P}_{6x}(q) = 0, \tag{191}$$

shows that $\mathrm{HS}_{\alpha=0}$ and SK admit a similar set of two soliton solutions:

$$\begin{aligned} &u_2 = 2\partial_x^2 \ln[1 + e^{\theta_1} + e^{\theta_2} + A_{12}^{\mathrm{SK}} e^{\theta_1+\theta_2}], \\ &A_{12}^{\mathrm{SK}} = \frac{(k_1-k_2)^2(k_1^2 - k_1k_2 + k_2^2)}{(k_1+k_2)^2(k_1^2 + k_1k_2 + k_2^2)}, \quad \theta_i = k_i x + \omega_i t + \eta_i \end{aligned} \tag{192}$$

with $\omega_i = k_i^3$ for SK and $\omega_i = k_i^{-1}$ for $\mathrm{HS}_{\alpha=0}$.

14.2 The AKNS Shallow Water Wave Equation

If $\beta^2 \neq 1$ we may decouple (181) into a pair of $\mathcal{P}$-conditions, by the introduction of an *auxiliary* variable t_3 and an *auxiliary* $\mathcal{P}$-condition of primary KdV type:

$$\mathcal{P}_{x,t_3}(q) - \mathcal{P}_{4x}(q) = 0. \tag{193}$$

This decoupling leads to a $\mathcal{P}$-representation which comprises (193) and the following $1+2$ dimensional condition:

$$2\mathcal{P}_{3x,t}(q) - (1-\beta)\mathcal{P}_{t,t_3}(q) + \alpha(1+\beta)\mathcal{P}_{x,t}(q) - (1+\beta)\mathcal{P}_{2x}(q) = 0. \tag{194}$$

Both conditions admit a common set of KdV like two soliton solutions:

$$\begin{aligned} &q_2 = 2\ln\big[1 + e^{\theta_1} + e^{\theta_2} + A_{12}^{\mathrm{KdV}} e^{\theta_1+\theta_2}\big], \\ &A_{12}^{\mathrm{KdV}} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \quad \theta_i = k_i x + k_i^3 t_3 + \frac{k_i}{\alpha + k_i^2} t + \eta_i, \end{aligned} \tag{195}$$

iff $\beta = 2$.

Equation (180) taken with $\beta = 2$ is the potential version of an equation for $u = q_{2x} = \frac{1}{2}V_x$, which is known as the *AKNS* shallow water wave equation [1]:

$$\mathrm{AKNS}(u) \equiv u_{2x,t} - u_x + \alpha u_t + 4uu_t + 2u_x \int_{-\infty}^{x} u_t dx' = 0. \tag{196}$$

To clarify the link between KdV and AKNS, we may look for a spectral formulation of the latter, by application of the direct procedure to its $\mathcal{P}$-representation:

$$E_{\mathrm{KdV}}(q) \equiv \mathcal{P}_{x,t_3}(q) - \mathcal{P}_{4x}(q) = 0, \tag{197}$$

$$E_\alpha(q) \equiv 2\mathcal{P}_{3x,t}(q) + \mathcal{P}_{t,t_3}(q) + 3\alpha\mathcal{P}_{x,t}(q) - 3\mathcal{P}_{2x}(q) = 0. \tag{198}$$

It is easy to verify that the two λ-dependent $\mathcal{Y}$-constraints (43, 44), into which the invariance condition on $\mathrm{E}_{\mathrm{KdV}}(q)$ can be decoupled, transform the other invariance condition:

$$E_\alpha(w+v) - E_\alpha(w-v) = 0 \tag{199}$$

into the x-derivative of the following $\mathcal{Y}$-constraint (of weight 1):

$$\mathcal{Y}_{2x,t}(v,w) - \mathcal{Y}_x(v) + (\alpha + 3\lambda)\mathcal{Y}_t(v) = 0. \tag{200}$$

The integrability of the linear counterpart to system (43, 200):

$$\phi_{2x} + q_{2x}\phi = \lambda\phi, \tag{201}$$

$$\phi_{2x,t} + (q_{2x} + \alpha + 3\lambda)\phi_t + (2q_{x,t} - 1)\phi_x = 0 \tag{202}$$

is subject to a condition on q which coincides with (196) expressed in terms of q:

$$\mathrm{AKNS}(q_{2x}) \equiv q_{4x,t} - q_{3x} + 4q_{2x}q_{2x,t} + 2q_{x,t}q_{3x} + \alpha q_{2x,t} = 0. \tag{203}$$

A spectral formulation of AKNS is therefore constituted by KdV's eigenvalue equation (201) and the following $1+1$ dimensional evolution equation (obtained from (202) by elimination of the λ-dependent term):

$$4\phi_{2x,t} + (4q_{2x} + \alpha)\phi_t + (5q_{x,t} - 1)\phi_x = 0. \tag{204}$$

The above analysis of (180) shows the relevance of the $\mathcal{P}$-basis decomposition method. It enabled us to identify two soliton equations, and to relate them to known families. In the first case (HS) the $\mathcal{P}$-representation is quite obvious, but the link with the SK-family becomes only apparent at the level of the corresponding linear system. In the second case (AKNS) the disclore of a $\mathcal{P}$-representation is less obvious, but hints directly at a link with KdV.

15 Conclusion

A first objective of this paper was to show that the three questions formulated in Sect. 1 can be answered positively by means of elementary considerations, related to basic invariance properties. The answers were obtained in a natural order, with the only input of Miura's original observation (link between KdV and mKdV).

The length of the paper was motivated by another concern: to present a self-contained introduction to soliton theory, based on standard concepts of linear algebra and a systematic use of exponential (i.e. partitional) polynomials. The main part was to show that there exists a simple and direct way leading from combinatorial aspects of logarithmically linearizable NLPDE's to the characterization of fundamental hierarchies of soliton equations. $\mathcal{Y}$-constraints and $\mathcal{P}$-conditions turned out to be appropriate tools to represent these hierarchies. They are in one to one correspondence with the more familiar bilinear expressions of Hirota, known to facilitate the search of explicit solutions the NLPDE's. An advantage of reformulating the Hirota approach in terms of $\mathcal{Y}$- and $\mathcal{P}$-polynomials lies in the clear view it provides of the linear multidimensional spaces to which members of the KP hierarchy (and related hierarchies) belong. These partitional polynomials are also a unifying element between soliton equations and their spectral formulation, the crux being the logarithmic transformation which linearizes the pivotal $\mathcal{Y}$-constraints.

Though many of the equations presented in this work are reformulations of known results, such as the various KP, mKP, BKP and DKP equations for which corresponding bilinear expressions have been listed by Jimbo and Miwa [22]), it is felt that this elementary and integrated approach is a novel and explanatory element worth to be reported in detail.

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