An elementary approach to hierarchies of soliton equations

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In this paper we present a systematic and elementary construction of the Hirota equations (in an equivalent formulation) that make up the bilinear KP and modified KP hierarchies. Our construction leads to a natural gradation of the Hirota equations of a given weight in each bilinear hierarchy, providing useful insight into their mutual relationship.

KEYWORDS: Hirota bilinear forms, KP hierarchy, Zakharov-Shabat formulation, Bell polynomials, Faà di Bruno iterates

1. Introduction

The Kadomtsev-Petviashvili (KP) equation[1] (where $u_{px} = \partial_x^p u$)

$$(u_t - u_{3x} - 6uu_x)_x - 3u_{2y} = 0 \tag{1}$$

is perhaps the most famous and fundamental soliton system in 2+1 dimensions. It arises in the study of shallow water waves and comprises two well-known 1+1 dimensional soliton systems as particular reductions : the Korteweg-de Vries and Boussinesq equations. An essential integrability feature of this nonlinear partial differential equation (NLPDE) is the fact that it can be derived from a "potential" equation,

$$u_t - u_{3x} - 6uu_x = 3U_y, \quad U_x = u_y, \tag{2}$$

which is the compatibility condition $L_t - M_y = [M, L]$ of a pair of linear differential operators L, M,

$$L = \partial_x^2 + u, \quad M = 4\partial^3 + 6u\partial_x + 3(u_x + U). \tag{3}$$

These operators can be used to define a set of linear differential equations

$$\psi_y = L\psi, \quad \psi_t = M\psi, \tag{4}$$

which we shall refer to as the Zakharov-Shabat formulation of the KP equation. A crucial feature of the potential KP equation (2) is that it can be derived from a quadratic, so-called

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Hirota bilinear equation,[2]

$$\left(\mathrm{D}_x\mathrm{D}_t - \mathrm{D}_x^4 - 3\mathrm{D}_y^2\right)\tau\cdot\tau = 0, \tag{5}$$

by means of the dependent variable transformation

$$u = 2\partial_x^2 \log\tau, \quad U = 2\partial_x\partial_y \log\tau. \tag{6}$$

The combinations of Hirota D-operators used in this reformulation of equation (1) are defined by their (bilinear) action on a pair of functions :

$$\mathrm{D}_x^r\mathrm{D}_t^s\ f\cdot g = (\partial_x - \partial_{x'})^r(\partial_t - \partial_{t'})^s f(x,t)g(x',t')|_{x'=x,t'=t}, \tag{7}$$

(or by the obvious extension to a product of operators involving an arbitrary number of independent variables). Note that $\mathrm{D}_x^r\mathrm{D}_t^s\ f\cdot g = (-1)^{r+s}\ \mathrm{D}_x^r\mathrm{D}_t^s\ g\cdot f$ and hence, that any odd degree combination of D operators acting on a pair of identical functions is trivially zero.

An equally fundamental integrability feature of the KP equation – in any of its forms – is that it is part of an infinite hierarchy of integrable NPLDEs of increasing order in x, commonly referred to as the KP-hierarchy.[3] All equations contained in this hierarchy allow for Zakharov-Shabat formulations, compatible with (4), and they can all be related to bilinear equations of Hirota-type.[3–5] In fact, it is the very existence of such an associated hierarchy of bilinear integrable equations that is central to the overarching theory known as Sato theory,[5,6] in which the KP time-evolution is thought of as a dynamical system on an infinite dimensional Grassmann manifold (the Sato Grassmannian). The central concept in this theory is that of the so-called *tau function* – expressible in terms of the Plücker coordinates on the Grassmannian – which satisfies an infinite set of Hirota bilinear equations that are intimately related to the Plücker relations for the Grassmannian. These bilinear equations for the tau function then give rise to the equations in the KP hierarchy mentioned before, through the use of dependent variable transformations like (6).

The description of the KP hierarchy offered by Sato theory is not the only one that gives accesss to the evolution equations making up the hierarchy. The so called "dressing"[7] or $\bar{\partial}$ methods,[8] or the algebro-geometric approach à la Krichever[9] all yield algebraic procedures that allow for the derivation of the KP equation and its higher order partners, essentially through the construction of linear formulations for these equations. The power of Sato-theory however lies in the introduction of the tau function as the cornerstone of the theory, not only as a means to describe explicit solutions to the equations in the hierarchy, but ultimately because of the bridge it provides between the KP hierarchy and the algebro-geometric object - the Sato Grassmannian - underpinning its integrability.

It is by now common knowledge that the Plücker relations on the Grassmannian give rise to Hirota bilinear equations (of which (5) is the simplest example), the full set of which, in turn, offers the following operational definition of a tau function : it is the object that satisfies all equations in the "bilinear" KP hierarchy, i.e., in the set of *all possible* Hirota-bilinear equations that can be obtained from the Plücker relations for the Grassmannian. It is worth emphasizing the difference which is implied by the use of the terms "KP hierarchy" and "bilinear KP hierarchy" : the former refers to the infinite set of commuting flows for the original fields u and U of equation (2), the latter however refers to the infinite set of Hirota equations that define a tau function in the above mentioned way.

It is striking that although the KP hierarchy is constituted by a single evolution equation for every order in x (larger than or equal to 4), in its bilinear counterpart there will generally be several independent, non-trivial bilinear equations of a given order in x. Apart form their obvious theoretical importance, these bilinear equations can actually serve a practical purpose as well : they can be used to construct recursion operators and bi-Hamiltonian formulations for 1+1 dimensional integrable reductions of the KP hierarchy, through the use of a so-called canonical bilinear forms.[10] For this to succeed however, one needs a complete set of (reduced) bilinear equations up to a particular weight. Sato theory, in any of its formulations,[5,11] offers certain means to obtain all such bilinear equations, most notably through the use of the so-called KP bilinear identity.[5] However, whereas the bilinear identity is indeed a generating function for all bilinear KP equations, to sift through the immense and (for increasing orders) rapidly proliferating number of equations it contains, trying to pick out the exact number of independent bilinear equations at a given order, is a highly inefficient way of constructing the bilinear KP hierarchy. There exists a procedure[12,13] that performs this task in a rather elegant way by recursively constructing bilinear equations from a sequence of Young diagrams.[12] However, this procedure is far from elementary as it relies on the theory of supersymmetric functions.

A fairly efficient procedure to construct the KP hierarchy (based on the concept of Darboux covariance[14]) was presented in.[15] The method explained here is to be regarded as an equally elementary yet more powerful alternative to that procedure. It is based on the systematic use of generalized Bell polynomials[16] – also sometimes referred to as Faà di Bruno iterates[17] – whose relevance to the study of integrable hierarchies was already pointed out by several authors.[18–21] The generalized Bell polynomials used in this paper, the so-called $\mathcal{Y}$ and $\mathcal{P}$-polynomials, are defined with respect to an infinite set of independent variables of integer weight. They will be used to generate linear spaces of equations to which the subsequent members of the bilinear KP and modified KP (mKP) hierarchies belong, thereby opening a simple and direct route to their identification.

In order to introduce the method we take the first two members of the mKP-hierarchy – which yield the linear formulation of the KP equation – and explain how a systematic construction of these mKP equations in Hirota bilinear form can be used to generate bilinear equations in the KP hierarchy, i.e. : they are the only two "$\mathcal{Y}$-constraints" (i.e.: constraints on a pair of dimensionless variables v and w, *linear* with respect to a basis of $\mathcal{Y}$-polynomials) of weights 2 and 3, that are mutually compatible under a single (nontrivial) condition on the variable $q = w - v$. This compatibility condition is found to be satisfied if q solves an NLPDE which is *linear* with respect to a set of $\mathcal{P}$-polynomials (of weight 4). This "$\mathcal{P}$-equation" is identified as the lowest member of the bilinear KP hierarchy. We then go on to show that higher order members of both the bilinear KP and mKP hierarchies can be obtained by a systematic identification of parameter families of $\mathcal{Y}$-constraints of increasing weight, mutually compatible under certain (nonlinear) conditions on q. A remarkable feature of these conditions is that they turn out to be always expressible as particular $\mathcal{P}$-equations (or x-derivatives thereof) which then provide higher weight members of the bilinear KP hierarchy.

The fact that the constraints we obtain are linear combinations of $\mathcal{Y}$ or $\mathcal{P}$ polynomials, demonstrates in an elementary way that the Hirota equations of given weight in the KP and mKP hierarchies form linear spaces, although they are essentially quadratic relations between derivatives of tau functions (with hindsight, one might claim that it probably is this observation that led Mikio Sato to a description of the KP hierarchy in terms of Plücker relations[22] ; Sato himself refers to similar calculations in[23]).

We list bases for all spaces of bilinear mKP equations up to weight 9 and for all bilinear KP equations up to weight 10 (the weight 9 and 10 Hirota equations are given in appendix). These bases exhibit a striking feature of the bilinear KP hierarchy : at every weight level greater than 3 there exists a Hirota equation that only involves variables of weights up to 4 (i.e., x, t_2, t_3 and t_4), as of weight 5 there always are Hirota equations that only involve variables up to weight 5, etc... This internal gradation of the linear space associated with the bilinear KP equations of a particular weight, is a natural by-product of our procedure. A similar gradation is apparent in our construction of the bilinear mKP hierarchy as well.

2. $\mathcal{P}$ and $\mathcal{Y}$-polynomials

The link between the KP equation and its Zakharov-Shabat formulation can be made more transparent by introducing a dimensionless, potential, alternative to the original dependent variable u. If one associates a dimension to each independent variable as suggested by the invariance of equation (1) under the scaling transformation :

$$x \to \lambda x, y \to \lambda^2 y, t \to \lambda^3 t, u \to \lambda^{-2} u,$$

(choosing the dimension of x to be 1), it is natural to introduce a dimensionless variable q by setting $u = q_{2x}$. Thus, introducing new independent variables $y = t_2, t = \frac{1}{4}t_3$ as well, integration with respect to x yields the following potential version of equation (1) :

$$(4q_{x,t_3} - q_{4x} - 3q_{2x}^2)_x - 3q_{x,2t_2} = 0. \tag{8}$$

As was already pointed out in the introduction, this equation can be thought of as the compatibility condition of a pair of first order linear evolution equations :

$$\psi_{t_2} = L_2(q)\psi, \qquad L_2(q) = \partial_x^2 + q_{2x} \tag{9}$$

$$\psi_{t_3} = L_3(q)\psi, \qquad L_3(q) = \partial_x^3 + \frac{3}{2}q_{2x}\partial_x + \frac{3}{4}(q_{3x} + q_{x,t_2}) \tag{10}$$

(equivalent to the system (4) under the aforementioned transformation).

The potential equation (8) can obviously be derived from the following homogeneous "primary" NLPDE of weight 4 (the weight of each term being defined as minus its dimension),

$$\mathrm{KP}_4(q) \equiv 4q_{x,t_3} - q_{4x} - 3q_{2x}^2 - 3q_{2t_2} = 0, \tag{11}$$

which has the remarkable feature that it is equivalent to a quadratic expression in derivatives of the variable $\tau = e^{q/2}$, an equation that can be cast in the compact form

$$(4\mathrm{D}_x\mathrm{D}_{t_3} - 3\mathrm{D}_{t_2}^2 - \mathrm{D}_x^4)\tau\cdot\tau = 0, \tag{12}$$

in terms of the Hirota D-operators (7). Remark that the coefficients in this equation sum up to zero, a "peculiarity" that is due to a remarkable partitional balance between the linear and the nonlinear terms of equation (11) (note that the structure of the Hirota bilinear equation (12) mimicks the linear, dispersive part of the KP equation (8)). Because of this particular balance, the $\mathrm{KP}_4(q)$ equation can be reformulated as a linear combination of so-called $\mathcal{P}$-polynomials. These are defined as even order 2-dimensional Bell polynomials[16] in which all odd order derivatives of q have been set equal to zero :

$$\mathcal{P}_{m_1t_1,\ldots,m_rt_r}(q) = e^{-q(t_1,\ldots,t_r)}\partial_{t_1}^{m_1}\ldots\partial_{t_r}^{m_r}e^{q(t_1,\ldots,t_r)}\Big|_{q_{m_1t_1,\ldots,m_rt_r}=0 \text{ for} \sum_{i=1}^{r} m_i \text{ odd}} \tag{13}$$

In particular, it can be shown[24,25] that there exists a one-to-one correspondence between these polynomials and combinations of Hirota operators acting on a single function :

$$\mathrm{D}_{t_1}^{m_1}\cdots\mathrm{D}_{t_r}^{m_r}\ \tau\cdot\tau = \ \tau^2\ \mathcal{P}_{m_1t_1,\cdots,m_rt_r}(q = 2\log\tau), \tag{14}$$

when $m_1 + m_2 + \cdots + m_r$ is even. Hence, $\mathrm{KP}_4(q)$ can be re-expressed as a linear combination of $\mathcal{P}$-polynomials

$$\mathrm{KP}_4 \equiv 4\mathcal{P}_{x,t_3} - 3\mathcal{P}_{2t_2} - \mathcal{P}_{4x} = 0, \tag{15}$$

with

$$\mathcal{P}_{2t_2} = q_{2t_2},\ \mathcal{P}_{x,t_3} = q_{x,t_3}\ \text{ and }\ \mathcal{P}_{4x} = q_{4x} + 3q_{2x}^2. \tag{16}$$

A related and even more striking feature of the KP equation is the fact that its Zakharov-Shabat formulation (9, 10) can be regarded as the linearization of a pair of homogeneous constraints on two dimensionless variables v and w, which is linear with respect to a set of *binary* Bell polynomials $\mathcal{Y}$ defined as :

$$\mathcal{Y}_{m_1t_1,\dots,m_rt_r}(v,w)=e^{-y(t_1,\dots,t_r)}\partial_{t_1}^{m_1}\dots\partial_{t_r}^{m_r}e^{y(t_1,\dots,t_r)} \tag{17}$$

where

$$y_{m_1t_1,\dots,m_rt_r}=\begin{cases} v_{m_1t_1,\dots,m_rt_r} \text{ if } \sum_{i=1}^{r} m_i=\text{odd} \\ w_{m_1t_1,\dots,m_rt_r} \text{ if } \sum_{i=1}^{r} m_i=\text{even}\end{cases}.$$

Obviously, these $\mathcal{Y}$-polynomials reduce to $\mathcal{P}$-polynomials at $v=0$.

Indeed, using the formula[25]

$$\mathcal{Y}_{mx,nt}(v,v+q)|_{v=\ln\psi}=\psi^{-1}\sum_{r=0}^{m}\sum_{s=0}^{n}\binom{m}{r}\binom{n}{s}\mathcal{P}_{rx,st}(q)\psi_{(m-r)x,(n-s)t}(q) \tag{18}$$

it is easily verified that the "$\mathcal{Y}$-constraints":

$$\mathcal{Y}_{t_2}(v)-\mathcal{Y}_{2x}(v,w)\equiv v_{t_2}-(w_{2x}+v_x^2)=0 \tag{19}$$

$$\begin{aligned}\mathcal{Y}_{t_3}(v)-\tfrac{1}{4}\mathcal{Y}_{3x}(v,w)-\tfrac{3}{4}\mathcal{Y}_{x,t_2}(v,w)\equiv v_{t_3} &-\frac{1}{4}(v_{3x}+3v_xw_{2x}+v_x^3)\\ &-\frac{3}{4}(w_{x,t_2}+v_xv_{t_2})=0\end{aligned} \tag{20}$$

are mapped onto the linear equations (9, 10) if we take

$$w=v+q,\ \ v=\ln\psi. \tag{21}$$

As was the case for the $\mathcal{P}$-polynomials, these $\mathcal{Y}$ polynomials are also in one-to-one correspondence with the action of the Hirota D-operators, this time on a general pair of functions[24] :

$$(\tau'\tau)^{-1}\ \mathrm{D}_{t_1}^{m_1}\cdots\mathrm{D}_{t_r}^{m_r}\ \tau'\cdot\tau\equiv\ \mathcal{Y}_{m_1t_1,\cdots,m_rt_r}(v=\ln\frac{\tau'}{\tau},w=\ln\tau'\tau). \tag{22}$$

Hence, the $\mathcal{Y}$-constraints (19, 20) are equivalent to the (bilinear) Hirota equations :

$$(\mathrm{D}_{t_2}-\mathrm{D}_x^2)\tau'\cdot\tau=0 \tag{23}$$

$$(4\mathrm{D}_{t_3}-\mathrm{D}_x^3-3\mathrm{D}_x\mathrm{D}_{t_2})\tau'\cdot\tau=0. \tag{24}$$

A point which deserves to be underlined is that equation (19) is a general $\mathcal{Y}$-constraint (of homogeneous weight, with coefficients that sum up to zero for an appropriate scaling of t_2) which contains all possible $\mathcal{Y}$-polynomials of weight 2 that can be defined with respect to the variables $t_1=x,t_2,t_3\dots$. Similarly, it can be easily seen that equation (20) is actually the only $\mathcal{Y}$-constraint of weight 3 that is compatible with (19) under a single, nontrivial, constraint on the variable q $(= w - v)$. Indeed, the most general homegeneous $\mathcal{Y}$-constraint of weight 3, with coefficients that sum up to zero (up to rescaling of t_3), is given by the following one parameter family of $\mathcal{Y}$-constraints (for the parameter α) :

$$\mathcal{Y}_{t_3} - \alpha\mathcal{Y}_{3x} - (1-\alpha)\mathcal{Y}_{x,t_2} = 0, \tag{25}$$

The linear counterparts of eqs. (19, 25) (obtained by means of the transformation (21) and formula (18))

$$\psi_{t_2} = \psi_{2x} + q_{2x}\psi \tag{26}$$

$$\psi_{t_3} = \psi_{3x} + (1+2\alpha)q_{2x}\psi_x + (1-\alpha)(q_{3x} + q_{x,t_2})\psi \tag{27}$$

are mutually compatible iff :

$$\begin{aligned}\psi_{t_2,t_3} - \psi_{t_3,t_2} = (4\alpha - 1)q_{3x}\psi_{2x} + (1-4\alpha)q_{2x,t_2}\psi_x \\ +[q_{2x,t_3} - \alpha q_{5x} - (1-\alpha)q_{x,2t_2} - (1+2\alpha)q_{2x}q_{3x}]\psi = 0\end{aligned} \tag{28}$$

i.e. iff $\alpha = \frac{1}{4}$, if q is to obey a nontrivial condition – in this case equation (8). Remark that the value of α obtained here is in exact agreement with the Zakharov-Shabat formulation (4).

The bilinear equations (23, 24) are known to be the first two members of the mKP-hierarchy[5] and we shall refer to them (or to the equivalent $\mathcal{Y}$-constraints (19) and (20)) as mKP_2 and mKP_3 respectively. As was seen above, these equations were obtained by selecting that particular pair of homogeneous $\mathcal{Y}$-constraints of lowest weight (with coefficients summing up to zero) that is compatible under a single nonlinear condition on $q = w - v$. Here, this constraint takes the form of a 3-dimensional NLPDE. Furthermore, it is found that the lowest degree member of the bilinear KP-hierarchy (12) is obtained by observing that this 3-dimensional NLPDE is a nonlinear condition on q of weight 5 and of "odd" character (as each of its terms contains an odd number of derivatives)

$$\begin{aligned}C_5^{\text{odd}}(t_{i\leqslant 3}) &\equiv q_{2x,t_3} - \tfrac{1}{4}q_{5x} - \tfrac{3}{4}q_{x,2t_2} - \tfrac{3}{2}q_{2x}q_{3x} \\ &\equiv \left[\mathcal{P}_{x,t_3}(q) - \tfrac{3}{4}\mathcal{P}_{2t_2}(q) - \tfrac{1}{4}\mathcal{P}_{4x}(q)\right]_x = 0\end{aligned} \tag{29}$$

which can be expressed as the x-derivative of a homogeneous linear combination of weight 4 $\mathcal{P}$-polynomials (with coefficients adding up to zero) – a so-called "$\mathcal{P}$-equation" of weight 4 : the KP_4 equation (15).

In view of these observations it seems natural to examine whether, similarly, higher order members of the mKP-hierarchy can be obtained through identification of homogeneous $\mathcal{Y}$-constraints of increasing weight $r > 3$ that are compatible with (all) lower weight constraints on $q(x, t_2, \ldots, t_r)$, and especially : to investigate whether these nonlinear conditions naturally take the form of $\mathcal{P}$-equations, which will then yield the higher order members of the bilinear KP hierarchy.

3. mKP-equations of weight r ⩽ 4 and related KP-equations

At weight 4, the most general homogeneous $\mathcal{Y}$-constraint takes the form of the 3-parameter family (where $\mathcal{Y}_{mx,nt}$ stands for $\mathcal{Y}_{mx,nt}(v,w)$) :

$$\mathcal{Y}_{t_4} = \alpha_1 \mathcal{Y}_{4x} + \alpha_2 \mathcal{Y}_{x,t_3} + \alpha_3 \mathcal{Y}_{2x,t_2} + (1-\alpha_1-\alpha_2-\alpha_3)\mathcal{Y}_{2t_2}. \tag{30}$$

Taking into account eqs.(9, 10), equation (30) can be mapped onto a linear equation for the variable $\psi = e^v$, which takes the form of a linear evolution equation for t_4:

$$\begin{aligned}\psi_{t_4} = \psi_{4x} &+ (2+4\alpha_1-\tfrac{1}{2}\alpha_2)q_{2x}\psi_{2x} + [(2-2\alpha_1+\tfrac{1}{4}\alpha_2)q_{3x} \\ &+ (\tfrac{3}{4}\alpha_2+2\alpha_3)q_{x,t_2}]\psi_x + [(1-\tfrac{1}{4}\alpha_2)q_{4x} + \alpha_2 q_{x,t_3} \\ &+ (1-\alpha_1-\tfrac{1}{4}\alpha_2-\alpha_3)q_{2x,t_2} + (1-\alpha_1-\alpha_2-\alpha_3)q_{2t_2} \\ &+ (1+2\alpha_1-\alpha_2)q_{2x}^2]\psi.\end{aligned} \tag{31}$$

Compatibility of this equation with equation (9) is easily seen to be subject to two non-trivial (nonlinear) conditions on q, of weights 5 and 6 respectively, iff:

$$\alpha_2 = 8\alpha_1, \ \alpha_3 = \frac{1}{2} - 3\alpha_1. \tag{32}$$

This restricts equation (31) to :

$$\begin{aligned}\psi_{t_4} = L_4(q;\alpha_1)\psi, \ L_4(q;\alpha_1) = \partial_x^4 &+ 2q_{2x}\partial_x^2 + (2q_{3x}+q_{x,t_2})\partial_x + 2\alpha_1 \mathrm{KP}_4 \\ &+ q_{4x} + \tfrac{1}{2}q_{2x,t_2} + \tfrac{1}{2}q_{2t_2} + q_{2x}^2,\end{aligned} \tag{33}$$

and yields compatibility conditions of the form:

$$C_5^{\mathrm{odd}}(t_{i\leqslant 3}) = 0, \quad C_6^{\mathrm{even}}(t_{i\leqslant 3}) + C_6^{\mathrm{odd}}(t_{i\leqslant 4}) = 0, \tag{34}$$

in which

$$C_5^{\mathrm{odd}}(t_{i\leqslant 3}) = 4\alpha_1(\mathrm{KP}_4)_x, \quad C_6^{\mathrm{even}}(t_{i\leqslant 3}) = 2\alpha_1(\mathrm{KP}_4)_{2x}, \tag{35}$$

for an expression $C_6^{\mathrm{odd}}(t_{i\leqslant 4})$ which, under the constraint implied by the KP_4 equation (15), can be shown to reduce to the x-derivative of a $\mathcal{P}$-equation of weight 5 (whose coefficients add up to zero) :

$$\mathrm{KP}_5 \equiv 3\mathcal{P}_{x,t_4}(q) - 2\mathcal{P}_{t_2,t_3}(q) - \mathcal{P}_{3x,t_2}(q) = 0. \tag{36}$$

On the other hand, the compatibility of equation (33) with equation (10) is found to be subject to three nonlinear conditions on q (of weights 5, 6 and 7 resp.), all of which are identically satisfied under the two KP-conditions (15) and (36) we already obtained. Hence, we conclude that each constraint of the 1-parameter family

$$\mathcal{Y}_{t_4} = \frac{1}{2}\mathcal{Y}_{2x,t_2} + \frac{1}{2}\mathcal{Y}_{2t_2} + \alpha_1 \left[\mathcal{Y}_{4x} + 8\mathcal{Y}_{x,t_3} - 3\mathcal{Y}_{2x,t_2} - 6\mathcal{Y}_{2t_2}\right] \tag{37}$$

(obtained from (31) under condition (32)) together with mKP_2 and mKP_3, constitute a set of homogeneous $\mathcal{Y}$-constraints, mutually compatible if q satisfies the KP_4 and KP_5 equations. Alternatively, we can thus say that possible (i.e. "compatible") bilinear mKP equations of weight 4 form a 2-dimensional linear space of $\mathcal{Y}$-constraints. As a basis for this linear space we distinguish two particular mKP_4 equations :

$$\mathrm{mKP}_4(t_{i\leqslant 3}) \equiv \mathcal{Y}_{4x} + 8\mathcal{Y}_{x,t_3} - 3\mathcal{Y}_{2x,t_2} - 6\mathcal{Y}_{2t_2} = 0 \tag{38}$$

$$\mathrm{mKP}_4(t_{i\leqslant 4}) \equiv \mathcal{Y}_{t_4} - \frac{1}{2}\mathcal{Y}_{2x,t_2} - \frac{1}{2}\mathcal{Y}_{2t_2} = 0, \tag{39}$$

the first one of which only involves independent variables up to dimension 3 and hence is nothing but a differential consequence of the previously obtained constraints mKP_2 and mKP_3. Accordingly, one can also interpret the bilinear mKP_2 and mKP_3 equations (23) and (24) as generating two 1-dimensional linear spaces of $\mathcal{Y}$-constraints of weights 2 and 3 resp., with basis vectors (19) and (20).

It is remarkable that all compatibility conditions (of weight 7 and 8) associated with equations (9, 10, 33) can be re-written in Hirota bilinear form, either directly or as the x-derivative of a bilinear equation. The higher weight members of the bilinear KP-hierarchy thus obtained only involve $t_{i\leqslant 4}$ variables, and therefore necessarily correspond to differential consequences of KP_4 and KP_5. For example, collecting terms of "even" and of "odd" character that appear in the compatibility condition of weight 7 that arises between equations (10) and (33), one finds two nonlinear conditions, $C_7^{\mathrm{even}}(t_{i\leqslant 4})$ and $C_7^{\mathrm{odd}}(t_{i\leqslant 4})$. Both are found to be re-expressible (by means of KP_4 and KP_5) as a linear combination of $\mathcal{P}$-polynomials (for the "even" one) or as the x-derivative of such a combination (for the "odd" one):

$$C_7^{\mathrm{even}}(t_{i\leqslant 4})\Big|_{\mathrm{KP}_4,\mathrm{KP}_5} = \tfrac{3}{2}[\mathcal{P}_{3x,t_4} - 2\mathcal{P}_{2x,t_2,t_3} + \mathcal{P}_{x,3t_2}] \tag{40}$$

$$C_7^{\mathrm{odd}}(t_{i\leqslant 4})\Big|_{\mathrm{KP}_4,\mathrm{KP}_5} = \tfrac{1}{4}(\alpha_1 + \tfrac{1}{12})[\mathcal{P}_{6x} - 9\mathcal{P}_{2x,2t_2} + 4\mathcal{P}_{3x,t_3} - 32\mathcal{P}_{2t_3} + 36\mathcal{P}_{t_2,t_4}]_x, \tag{41}$$

(where $\mathcal{P}_{mx,nt}$ stands for $\mathcal{P}_{mx,nt}(q)$). The corresponding $\mathcal{P}$-equations:

$$\mathrm{KP}_6(t_{i\leqslant 4}) \equiv \mathcal{P}_{6x} - 9\mathcal{P}_{2x,2t_2} + 4\mathcal{P}_{3x,t_3} - 32\mathcal{P}_{2t_3} + 36\mathcal{P}_{t_2,t_4} = 0 \tag{42}$$

$$\mathrm{KP}_7(t_{i\leqslant 4}) \equiv \mathcal{P}_{3x,t_4} - 2\mathcal{P}_{2x,t_2,t_3} + \mathcal{P}_{x,3t_2} = 0 \tag{43}$$

yield two members (of weights 6 and 7 resp.) of the bilinear KP-hierarchy.[5] Similarly, a degree 8 member of the bilinear KP hierarchy can be obtained from the compatibility condition of weight 8 that arises between two versions of (33), corresponding to two different values of the parameter α_1. Collecting the terms of "even" character, one obtains an expression $C_8^{\text{even}}(t_{i\leqslant 4})$ which, under the KP conditions (15, 36, 42), can be reduced to a linear combination of $\mathcal{P}$-polynomials of weight 8:

$$C_8^{\text{even}}(t_{i\leqslant 4})\Big|_{\text{KP}_4,\text{KP}_5,\text{KP}_6(t_{i\leqslant 4})} = \tfrac{1}{3}[\mathcal{P}_{8x} + 15\mathcal{P}_{4x,2t_2} - 20\mathcal{P}_{5x,t_3} + 64\mathcal{P}_{2x,2t_3} - 36\mathcal{P}_{4t_2} + 48\mathcal{P}_{x,2t_2,t_3} - 72\mathcal{P}_{2x,t_2,t_4}]. \tag{44}$$

We thus obtain the (unique) member of the KP-hierarchy of weight 8 which involves only variables of dimension up to 4 :

$$\text{KP}_8(t_{i\leqslant 4}) \equiv \mathcal{P}_{8x} + 15\mathcal{P}_{4x,2t_2} - 20\mathcal{P}_{5x,t_3} + 64\mathcal{P}_{2x,2t_3} - 36\mathcal{P}_{4t_2} + 48\mathcal{P}_{x,2t_2,t_3} - 72\mathcal{P}_{2x,t_2,t_4} = 0. \tag{45}$$

However, the expression $C_8^{\text{odd}}(t_{i\leqslant 4})$ which contains the terms of "odd" character reduces to the x-derivative of a linear combination of $\mathcal{P}$-polynomials of weight 7 that turns out to be equivalent to equation (43). Hence, no new $\mathcal{P}$-constraint arises from this condition.

It is worth pointing out that, as a by-product of the "reducibility" of these "even" and "odd" character expressions to $\mathcal{P}$-constraints, one also obtains the following curious relation between two seemingly unrelated combinations of $\mathcal{P}$-polynomials : from the reduction (40) of the expression $C_7^{even}(t_{i\leqslant 4})$, which originally takes a form proportional to $(\text{KP}_5)_{2x}$, one has that

$$\tfrac{1}{3}(\text{KP}_5)_{2x}\Big|_{\text{KP}_4,\text{KP}_5} = \text{KP}_7(t_{i\leqslant 4}). \tag{46}$$

4. mKP-equations of weight r ⩽ 5 and related KP-equations

At weight 5, the homogeneous $\mathcal{Y}$-constraints we have to consider form a 5-parameter family:

$$\mathcal{Y}_{t_5} = \alpha_1\mathcal{Y}_{5x} + \alpha_2\mathcal{Y}_{3x,t_2} + \alpha_3\mathcal{Y}_{2x,t_3} + \alpha_4\mathcal{Y}_{x,t_4} + \alpha_5\mathcal{Y}_{x,2t_2} + (1 - \sum_{i=1}^{5}\alpha_i)\mathcal{Y}_{t_2,t_3}. \tag{47}$$

Equation (47) can again be transformed, on account of mKP_2, mKP_3 and $\text{mKP}_4(t_{i\leqslant 4})$, into a linear evolution equation with respect to t_5. Compatibility of this equation with equation (9) is subject to three nontrivial conditions on q (of weights 5, 6 and 7 resp.) iff:

$$\alpha_3 = \frac{1}{4} - 4\alpha_1,\ \ \alpha_4 = 18\alpha_1 + 6\alpha_2,\ \ \alpha_5 = \frac{1}{2} - 3\alpha_1 - 3\alpha_2, \tag{48}$$

Under these restrictions, one obtains a linear evolution of weight 5 :

$$\psi_{t_5} = L_5(q;\alpha_1,\alpha_2)\psi \tag{49}$$

$$L_5(q;\alpha_1,\alpha_2) = \partial_x^5 + \tfrac{5}{2}q_{2x}\partial_x^3 + [\tfrac{15}{4}q_{3x} + \tfrac{5}{4}q_{x,t_2}]\partial_x^2 + [\tfrac{5}{4}q_{2x,t_2} - \tfrac{35}{4}q_{2t_2} + \tfrac{25}{2}q_{x,t_3} - \tfrac{15}{2}q_{2x}^2 - 2\alpha_1\mathrm{KP}_4]\partial_x + \{\tfrac{15}{16}q_{5x} + \tfrac{5}{16}q_{x,2t_2} + \tfrac{5}{6}q_{3x,t_2} + \tfrac{5}{12}q_{t_2,t_3} + \tfrac{15}{8}q_{2x}q_{3x} + \tfrac{5}{4}q_{2x}q_{x,t_2} - (3\alpha_1+\alpha_2)[(\mathrm{KP}_4)_x - 2\mathrm{KP}_5]\} \tag{50}$$

which is compatible with equation (9), under the conditions KP_4 and KP_5, iff q satisfies an additional $\mathcal{P}$-equation of weight 6 (the x-derivative of which is obtained from the reduction of the "odd" character terms that appear in the condition of weight 7) :

$$\mathrm{KP}_6(t_{i\leqslant 5}) \equiv 144\mathcal{P}_{x,t_5} + \mathcal{P}_{6x} - 45\mathcal{P}_{2x,2t_2} - 20\mathcal{P}_{3x,t_3} - 80\mathcal{P}_{2t_3} = 0. \tag{51}$$

In fact, we find that all members of the 2-parameter family:

$$\mathcal{Y}_{t_5} = \tfrac{1}{4}\mathcal{Y}_{2x,t_3} + \tfrac{3}{4}\mathcal{Y}_{t_2,t_3} + \tfrac{1}{2}\mathcal{Y}_{x,2t_2} - \tfrac{1}{2}\mathcal{Y}_{x,t_4} + \alpha_1(\mathcal{Y}_{5x} - 4\mathcal{Y}_{2x,t_3} - 12\mathcal{Y}_{t_2,t_3} - 3\mathcal{Y}_{x,2t_2} + 18\mathcal{Y}_{x,t_4}) + \alpha_2(\mathcal{Y}_{3x,t_2} - 4\mathcal{Y}_{t_2,t_3} - 3\mathcal{Y}_{x,2t_2} + 6\mathcal{Y}_{x,t_4}) \tag{52}$$

together with mKP_2, mKP_3 and $\mathrm{mKP}_4(t_{i\leqslant 4})$, constitute a set of homogeneous $\mathcal{Y}$-constraints, mutually compatible under the three KP-conditions KP_4, KP_5 and $\mathrm{KP}_6(t_{i\leqslant 5})$. Alternatively, one can think of the bilinear equations of weight 5 in the mKP hierarchy as forming a 3-dimensional linear space, a basis for which we choose as :

$$\mathrm{mKP}_5(t_{i\leqslant 3}) \equiv \mathcal{Y}_{5x} - 3\mathcal{Y}_{3x,t_2} - 4\mathcal{Y}_{2x,t_3} + 6\mathcal{Y}_{x,2t_2} = 0 \tag{53}$$

$$\mathrm{mKP}_5(t_{i\leqslant 4}) \equiv \mathcal{Y}_{3x,t_2} - 3\mathcal{Y}_{x,2t_2} - 4\mathcal{Y}_{t_2,t_3} + 6\mathcal{Y}_{x,t_4} = 0 \tag{54}$$

$$\mathrm{mKP}_5(t_{i\leqslant 5}) \equiv \mathcal{Y}_{t_5} - \frac{1}{16}\mathcal{Y}_{5x} - \frac{5}{16}\mathcal{Y}_{x,2t_2} - \frac{5}{8}\mathcal{Y}_{x,t_4} = 0, \tag{55}$$

such that the first two equations only involve t-variables that correspond to time evolutions governed by lower weight equations. Therefore, when expressed in terms of ψ and q, the equations (53) and (54) become mere differential consequences of mKP_2, mKP_3 and $\mathrm{mKP}_4(t_{i\leqslant 4})$.

Higher order members of the bilinear KP-hierarchy, of weights 7, 8 and 9 and involving only variables of weights up to 5, can be obtained from the nonlinear conditions of weight 8 and 9 that q has to satisfy for equation (49) to be compatible with equations (10) and (33) respectively. Each of these conditions contains two expressions of opposite "parity" which, under KP_4, KP_5 and $\mathrm{KP}_6(t_{i\leqslant 5})$, can either be reduced to a linear combination of $\mathcal{P}$-polynomials, or to the x-derivative of such a combination. The expressions of weight 8 produce one $\mathcal{P}$-equation of weight 7 and one of weight 8:

$$\mathrm{KP}_7(t_{i\leqslant 5}) \equiv \mathcal{P}_{5x,t_2} - 10\mathcal{P}_{2x,t_2,t_3} + 5\mathcal{P}_{3x,t_4} - 20\mathcal{P}_{t_3,t_4} + 24\mathcal{P}_{t_2,t_5} = 0 \tag{56}$$

$$\mathrm{KP}_8(t_{i\leqslant 5}) \equiv \mathcal{P}_{4t_2} - 4\mathcal{P}_{2x,2t_3} + 2\mathcal{P}_{x,2t_2,t_3} - 3\mathcal{P}_{2x,t_2,t_4} + 4\mathcal{P}_{3x,t_5} = 0, \tag{57}$$

whereas the weight 9 ones produce another (different) weight 8 $\mathcal{P}$-equation :

$$\begin{aligned}\widetilde{\mathrm{KP}}_8(t_{i\leqslant 5}) \equiv \mathcal{P}_{8x} - 10\mathcal{P}_{4x,2t_2} + 160\mathcal{P}_{2x,2t_3} - 35\mathcal{P}_{4t_2} + 300\mathcal{P}_{2t_4}\\ -96\mathcal{P}_{3x,t_5} - 320\mathcal{P}_{t_3,t_5} = 0\end{aligned} \tag{58}$$

as well as two $\mathcal{P}$-equations of weight 9, $\mathrm{KP}_9(t_{i\leqslant 4})$ and $\mathrm{KP}_9(t_{i\leqslant 5})$ (on account of the fact that the weight 9 expression of "even" character can be reduced to a 1-parameter family of $\mathcal{P}$-polynomials of that weight). These two KP_9 bilinear equations are given in the appendix.

Furthermore, four addditional KP-equations of weight 10, involving only $t_{i\leqslant 5}$ variables, can be obtained from the compatibility conditions that arise between two copies of equation (49) corresponding to different values of the parameters. These equations are also given in the appendix.

5. Further mKP-equations and KP-equations (of weights up to 10)

At weight 6, the possible $\mathcal{Y}$-constraints form a 9-parameter family, which comprises a 4-parameter (sub)family of constraints that are compatible with mKP_2 under a set of nontrivial conditions on q. These conditions are found to be satisfied, subject to KP_4, KP_5 and $\mathrm{KP}_6(t_{i\leqslant 5})$, iff q satisfies a particularly simple $\mathcal{P}$-equation of weight 7 (obtained from the "odd" character terms which appear in the condition of weight 8):

$$\mathrm{KP}_7(t_{i\leqslant 6}) \equiv \mathcal{P}_{2x,t_2,t_3} + \mathcal{P}_{t_3,t_4} - 2\mathcal{P}_{x,t_6} = 0. \tag{59}$$

All $\mathcal{Y}$-constraints belonging to the 4-parameter family that is compatible with mKP_2 are also found to be compatible with mKP_3, $\mathrm{mKP}_4(t_{i\leqslant 4})$ and $\mathrm{mKP}_5(t_{i\leqslant 5})$, if q satisfies the 4 KP conditions (15, 36, 51, 59). Hence, the bilinear mKP equations of weight 6 form a 5-dimensional linear space, spanned by the following 5 mKP_6-equations :

$$\mathrm{mKP}_6(t_{i\leqslant 3}) \equiv \mathcal{Y}_{6x} + 3\mathcal{Y}_{4x,t_2} + 12\mathcal{Y}_{3t_2} - 16\mathcal{Y}_{3x,t_3} = 0 \tag{60}$$

$$\mathrm{mKP}_6(t_{i\leqslant 4}) \equiv \mathcal{Y}_{4x,t_2} - 3\mathcal{Y}_{2x,2t_2} + 8\mathcal{Y}_{x,t_2,t_3} - 6\mathcal{Y}_{2x,t_4} = 0 \tag{61}$$

$$\widetilde{\mathrm{mKP}}_6(t_{i\leqslant 4}) \equiv \mathcal{Y}_{6x} - 9\mathcal{Y}_{4x,t_2} + 16\mathcal{Y}_{3x,t_3} + 64\mathcal{Y}_{2t_3} - 72\mathcal{Y}_{t_2,t_4} = 0 \tag{62}$$

$$\mathrm{mKP}_6(t_{i\leqslant 5}) \equiv 3\mathcal{Y}_{6x} + 5\mathcal{Y}_{4x,t_2} - 80\mathcal{Y}_{x,t_2,t_3} - 120\mathcal{Y}_{t_2,t_4} + 192\mathcal{Y}_{x,t_5} = 0 \tag{63}$$

$$\mathrm{mKP}_6(t_{i\leqslant 6}) \equiv 3\mathcal{Y}_{x,t_2,t_3} + \mathcal{Y}_{3x,t_3} + 2\mathcal{Y}_{2t_3} - 6\mathcal{Y}_{t_6} = 0 \tag{64}$$

Note that among these basic equations there are two equations that only involve times variables up to weight 4 ; mKP6($t_{i\leqslant 6}$) however defines a new linear evolution equation for the variable ψ.

The nonlinear conditions on q obtained by requiring compatibility of equation (64) with mKP_3, $\mathrm{mKP}_4(t_{i\leqslant 4})$ and $\mathrm{mKP}_5(t_{i\leqslant 5})$, are found to produce 21 additional bilinear KP-equations which only involve variables of weights up to 6. These comprise one equation of weight 8:

$$\mathrm{KP}_8(t_{i\leqslant 6}) \equiv \mathcal{P}_{4x,2t_2} - 4\mathcal{P}_{x,2t_2,t_3} - \mathcal{P}_{4t_2} - 12\mathcal{P}_{2t_4} + 16\mathcal{P}_{t_2,t_6} = 0, \tag{65}$$

as well as two equations at weight 9 and weight 10, seven equations at weight 11 and nine at weight 12 (the weight 9 and 10 KP-equations are given in appendix).

At weight 7 one finds a 6-parameter family of $\mathcal{Y}$-constraints that are compatible with all previous mKP-equations under a set of (nontrivial) conditions on q. Hence, these bilinear equations form a 7-dimensional linear space, spanned by :

$$\begin{aligned}\mathrm{mKP}_7(t_{i\leqslant 3}) \equiv \mathcal{Y}_{7x} &- \mathcal{Y}_{5x,t_2} + 8\mathcal{Y}_{3x,2t_2} + 12\mathcal{Y}_{x,3t_2} \\ &-4\mathcal{Y}_{4x,t_3} - 32\mathcal{Y}_{2x,t_2,t_3} + 16\mathcal{Y}_{2t_2,t_3} = 0\end{aligned} \tag{66}$$

$$\mathrm{mKP}_7(t_{i\leqslant 4}) \equiv \mathcal{Y}_{3x,2t_2} + \mathcal{Y}_{5x,t_2} - 4\mathcal{Y}_{2x,t_2,t_3} + 8\mathcal{Y}_{2t_2,t_3} - 6\mathcal{Y}_{3x,t_4} = 0 \tag{67}$$

$$\begin{aligned}\widetilde{\mathrm{mKP}}_7(t_{i\leqslant 4}) \equiv 3\mathcal{Y}_{3x,2t_2} &- 3\mathcal{Y}_{x,3t_2} - 4\mathcal{Y}_{4x,t_3} \\ &+16\mathcal{Y}_{x,2t_3} + 6\mathcal{Y}_{3x,t_4} - 18\mathcal{Y}_{x,t_2,t_4} = 0\end{aligned} \tag{68}$$

$$\begin{aligned}\mathrm{mKP}_7(t_{i\leqslant 5}) \equiv \mathcal{Y}_{7x} &+ 15\mathcal{Y}_{3x,2t_2} - 5\mathcal{Y}_{4x,t_3} - 45\mathcal{Y}_{2x,t_2,t_3} \\ &+30\mathcal{Y}_{2t_2,t_3} + 40\mathcal{Y}_{x,2t_3} - 36\mathcal{Y}_{2x,t_5} = 0\end{aligned} \tag{69}$$

$$\begin{aligned}\widetilde{\mathrm{mKP}}_7(t_{i\leqslant 5}) \equiv \mathcal{Y}_{7x} &+ 3\mathcal{Y}_{3x,2t_2} + 12\mathcal{Y}_{x,3t_2} - 4\mathcal{Y}_{4x,t_3} \\ &-27\mathcal{Y}_{2x,t_2,t_3} + 21\mathcal{Y}_{2t_2,t_3} + 30\mathcal{Y}_{t_3,t_4} - 36\mathcal{Y}_{t_2,t_5} = 0\end{aligned} \tag{70}$$

$$\mathrm{mKP}_7(t_{i\leqslant 6}) \equiv \mathcal{Y}_{4x,t_3} - 3\mathcal{Y}_{2t_2,t_3} - 4\mathcal{Y}_{x,2t_3} - 6\mathcal{Y}_{t_3,t_4} + 12\mathcal{Y}_{x,t_6} = 0 \tag{71}$$

$$\begin{aligned}\mathrm{mKP}_7(t_{i\leqslant 7}) \equiv 4\mathcal{Y}_{7x} &+ 42\mathcal{Y}_{3x,2t_2} + 42\mathcal{Y}_{x,3t_2} - 21\mathcal{Y}_{4x,t_3} - 126\mathcal{Y}_{2x,t_2,t_3} \\ &+105\mathcal{Y}_{2t_2,t_3} + 56\mathcal{Y}_{x,2t_3} + 42\mathcal{Y}_{t_3,t_4} - 144\mathcal{Y}_{t_7} = 0\end{aligned} \tag{72}$$

Among the conditions on q obtained by requiring compatibility of all mKP equations upto weight 7 one finds, besides the $\mathcal{P}$-equations (15, 36, 51) and (59) , one new $\mathcal{P}$-equation of weight 8

$$\mathrm{KP}_8(t_{i\leqslant 7}) \equiv 2\mathcal{P}_{x,2t_2,t_3} + 2\mathcal{P}_{2x,2t_3} + 3\mathcal{P}_{2x,t_2,t_4} + 3\mathcal{P}_{2t_4} + 4\mathcal{P}_{t_3,t_5} - 2\mathcal{P}_{t_2,t_6} - 12\mathcal{P}_{x,t_7} = 0, \tag{73}$$

as well as several higher weight KP-equations, involving only $t_{i\leqslant 7}$. Among these one finds one new equation of weight 9 ($\mathrm{KP}_9(t_{i\leqslant 7})$) and two equations of weight 10. These three equations are given in appendix.

At weight 8, one finds that the bilinear mKP equations of that weight form an 11-dimensional linear space, spanned by :

$$\mathrm{mKP}_8(t_{i\leqslant 3}) \equiv 128\mathcal{Y}_{t_2,2t_3} + 72\mathcal{Y}_{4t_2} - 96\mathcal{Y}_{x,2t_2,t_3} - 128\mathcal{Y}_{2x,2t_3} - 30\mathcal{Y}_{4x,2t_2} + 108\mathcal{Y}_{2x,3t_2} - 64\mathcal{Y}_{3x,t_2,t_3} - 8\mathcal{Y}_{5x,t_3} + 17\mathcal{Y}_{6x,t_2} + \mathcal{Y}_{8x} = 0 \tag{74}$$

$$\mathrm{mKP}_8(t_{i\leqslant 4}) \equiv 6\mathcal{Y}_{2t_2,t_4} - 3\mathcal{Y}_{4t_2} + 8\mathcal{Y}_{x,2t_2,t_3} - 12\mathcal{Y}_{2x,t_2,t_4} + \mathcal{Y}_{4x,2t_2} = 0 \tag{75}$$

$$\widetilde{\mathrm{mKP}}_8(t_{i\leqslant 4}) \equiv \mathcal{Y}_{6x,t_2} - 16\mathcal{Y}_{3x,t_2,t_3} + 6\mathcal{Y}_{4x,t_4} + 12\mathcal{Y}_{2x,3t_2} - 3\mathcal{Y}_{4x,2t_2} = 0 \tag{76}$$

$$\widehat{\mathrm{mKP}}_8(t_{i\leqslant 4}) \equiv \mathcal{Y}_{8x} + 9\mathcal{Y}_{6x,t_2} - 8\mathcal{Y}_{5x,t_3} - 18\mathcal{Y}_{4x,2t_2} + 36\mathcal{Y}_{2x,3t_2} + 36\mathcal{Y}_{4t_2} - 128\mathcal{Y}_{2x,2t_3} + 72\mathcal{Y}_{2t_2,t_4} = 0 \tag{77}$$

$$\mathrm{mKP}_8(t_{i\leqslant 5}) \equiv \mathcal{Y}_{8x} + 5\mathcal{Y}_{6x,t_2} + 40\mathcal{Y}_{3x,t_2,t_3} - 96\mathcal{Y}_{3x,t_5} - 160\mathcal{Y}_{2x,2t_3} + 30\mathcal{Y}_{4t_2} + 180\mathcal{Y}_{2t_2,t_4} = 0 \tag{78}$$

$$\widetilde{\mathrm{mKP}}_8(t_{i\leqslant 5}) \equiv 2\mathcal{Y}_{6x,t_2} - 15\mathcal{Y}_{4x,2t_2} + 40\mathcal{Y}_{3x,t_2,t_3} - 30\mathcal{Y}_{2x,3t_2} + 45\mathcal{Y}_{4t_2} + 288\mathcal{Y}_{x,t_2,t_5} - 240\mathcal{Y}_{x,t_3,t_4} - 90\mathcal{Y}_{2t_2,t_4} = 0 \tag{79}$$

$$\widehat{\mathrm{mKP}}_8(t_{i\leqslant 5}) \equiv 4\mathcal{Y}_{8x} + 26\mathcal{Y}_{6x,t_2} - 87\mathcal{Y}_{4x,2t_2} - 80\mathcal{Y}_{3x,t_2,t_3} + 234\mathcal{Y}_{2x,3t_2} - 352\mathcal{Y}_{2x,2t_3} - 120\mathcal{Y}_{x,2t_2,t_3} + 129\mathcal{Y}_{4t_2} + 198\mathcal{Y}_{2t_2,t_4} + 768\mathcal{Y}_{t_3,t_5} - 720\mathcal{Y}_{2t_4} = 0 \tag{80}$$

$$\mathrm{mKP}_8(t_{i\leqslant 6}) \equiv 3\mathcal{Y}_{6x,t_2} + 15\mathcal{Y}_{4x,2t_2} - 10\mathcal{Y}_{2x,3t_2} - 40\mathcal{Y}_{4t_2} + 90\mathcal{Y}_{2t_2,t_4} - 80\mathcal{Y}_{2x,t_6} - 192\mathcal{Y}_{x,t_2,t_5} + 240\mathcal{Y}_{x,t_3,t_4} - 80\mathcal{Y}_{t_2,t_6} + 60\mathcal{Y}_{2t_4} = 0 \tag{81}$$

$$\begin{aligned}\widetilde{\mathrm{mKP}}_8(t_{i\leqslant 6}) \equiv 3\mathcal{Y}_{8x} &+39\mathcal{Y}_{6x,t_2} - 225\mathcal{Y}_{4x,2t_2} + 30\mathcal{Y}_{2x,3t_2}\\ &+395\mathcal{Y}_{4t_2} - 810\mathcal{Y}_{2t_2,t_4} - 1680\mathcal{Y}_{x,t_3,t_4}\\ &+2016\mathcal{Y}_{x,t_2,t_5} + 352\mathcal{Y}_{3x,t_5} - 800\mathcal{Y}_{t_2,t_6}\\ &+1280\mathcal{Y}_{t_3,t_5} - 600\mathcal{Y}_{2t_4} = 0\end{aligned} \tag{82}$$

$$\begin{aligned}\mathrm{mKP}_8(t_{i\leqslant 7}) \equiv 2\mathcal{Y}_{8x} &+105\mathcal{Y}_{6x,t_2} + 245\mathcal{Y}_{4x,2t_2} - 770\mathcal{Y}_{2x,3t_2}\\ &+1288\mathcal{Y}_{3x,t_5} - 3080\mathcal{Y}_{x,t_2,t_5} - 1120\mathcal{Y}_{x,t_3,t_4}\\ &-1330\mathcal{Y}_{2t_2,t_4} - 8960\mathcal{Y}_{t_2,t_6} - 3360\mathcal{Y}_{t_3,t_5}\\ &+4340\mathcal{Y}_{2t_4} + 12640\mathcal{Y}_{x,t_7} = 0\end{aligned} \tag{83}$$

$$\begin{aligned}\mathrm{mKP}_8(t_{i\leqslant 8}) \equiv 3\mathcal{Y}_{8x} &+42\mathcal{Y}_{6x,t_2} + 21\mathcal{Y}_{4x,2t_2} + 700\mathcal{Y}_{3x,t_5} - 924\mathcal{Y}_{x,t_2,t_5}\\ &+168\mathcal{Y}_{x,t_3,t_4} - 378\mathcal{Y}_{2t_2,t_4} - 2352\mathcal{Y}_{t_2,t_6}\\ &-112\mathcal{Y}_{t_3,t_5} + 504\mathcal{Y}_{2t_4} + 4176\mathcal{Y}_{x,t_7} - 1848\mathcal{Y}_{t_8} = 0\end{aligned} \tag{84}$$

The compatibility conditions that arise among these equations and the lower weight mKP equations, are again satisfied under the former KP-conditions (15, 36, 51, 59, 73) supplemented by a new $\mathcal{P}$-equation of weight 9 ($\mathrm{KP}_9(t_{i\leqslant 8})$, which is given in the appendix.

It should be noted that the 5 bilinear equations of weight 9 listed in[5] (the highest weight equations in that list) are contained in the span of the basic KP_9 equations given in the appendix, were it not for a misprint in the last equation in.[5] The correct version being :

$$[\mathrm{D}_4(\mathrm{D}_1^5 + 20\mathrm{D}_1^2\mathrm{D}_3 + \mathbf{24}\mathrm{D}_5) + 15\mathrm{D}_1\mathrm{D}_2^2\mathrm{D}_4 - 60\mathrm{D}_1\mathrm{D}_8]\tau\cdot\tau = 0 \tag{85}$$

in which no $\mathrm{D}_3\mathrm{D}_6$-term is present and which corresponds to the combination:

$$\begin{aligned}-\frac{1}{4}\mathrm{KP}_9(t_{i\leqslant 4}) + \frac{1}{4}\mathrm{KP}_9(t_{i\leqslant 5}) &+ \frac{10}{3}\mathrm{KP}_9(t_{i\leqslant 6}) - \frac{2}{3}\widetilde{\mathrm{KP}}_9(t_{i\leqslant 6})\\ &-\frac{5}{21}\mathrm{KP}_9(t_{i\leqslant 7}) + \frac{20}{7}\mathrm{KP}_9(t_{i\leqslant 8}) = 0\end{aligned} \tag{86}$$

Among the higher weight bilinear KP-equations involving only $t_{i\leqslant 8}$ that can be derived from the mKP_8 equations, one obtains one additional equation of weight 10 ($\mathrm{KP}_{10}(t_{i\leqslant 8})$; see appendix).

Finally, the 15 basic equations that span the linear space of mKP equations of weight 9 are listed in appendix. Among the $\mathcal{P}$-equations that arise from the relevant compatibility conditions, one obtains one more KP equation ($\mathrm{KP}_{10}(t_{i\leqslant 9})$), completing the set of basis KP equations of weight 10 given in the appendix.

6. Conclusions

We believe that the method we presented here – whereby first all bilinear mKP equations of a given weight are constructed by requiring compatibility of a family of $\mathcal{Y}$-constraints and where, subsequently, from the compatibility conditions that appear at that level, bilinear KP equations of increasing weight but involving only a limited set of independent variables are obtained, as $\mathcal{P}$-equations – is a particularly efficient way to construct the equations that make up the bilinear KP and mKP hierarchies. This is especiallly true for the bilinear KP hierarchy, for which at a given weight the bulk of all bilinear equations can be expressed using just a small set of (lower weight) variables, and is therefore already obtained at an early stage in the construction. Obtaining the remaining equations, few in number, only requires calculating specific compatibility conditions. For example, the two KP equations (A·1),(A·2) of weight 9 that only involve variables up to weight 4 and 5 respectively, are already obtained when verifying the compatibility of mKP_5 with mKP_4 ; the two KP_9 equations (A·3),(A·4) involving variables up to weight 6 are obtained from the compatibility of mKP_6. Furthermore, it is important to note that, say in general for weight n, there only ever exists a single bilinear equation of type $\mathrm{KP}_n(t_{i\leq n-2})$ or $\mathrm{KP}_n(t_{i\leq n-1})$, as $\mathcal{P}_{x2,t_{n-2}}$ and $\mathcal{P}_{x,t_{n-1}}$ are the only weight n $\mathcal{P}$-polynomials involving variables up to t_{n-2} or t_{n-1}. Hence, in order to obtain these "higher" equations, it suffices to calculate just one compatibility condition at each remaining level (in general, for weight n : between mKP_{n-2} and mKP_3, and between mKP_{n-1} and mKP_2 respectively). This immediately ensures that all bilinear KP equations of a certain weight will be found after a finite (and comparatively small) number of steps. The process therefore does not require any advance knowledge of the exact number of bilinear equations of a given weight.

In fact, the problem of determining the number of Hirota equations of a certain type and weight was first posed by Sato, who conjectured that the number of bilinear mKP equations of weight n is equal to the number of partitions of $n-2$:

$$\#\{\mathrm{mKP}_n\} = p(n-2), \tag{87}$$

where the symbol $p(m)$ denotes the number of partitions of the integer m, if m is positive ; otherwise, we define $p(0) = 1$ and $p(m) = 0$ when m is negative. For example, in the appendix a basis of 15 mKP_9 equations is given, 15 being the number of partitions of 7. This conjecture was subsequently proven, in,[27] where yet another of Sato's conjectures – stating[26] that the number of bilinear KP equations of weight n is given by $p(n-1)$ – was also proven. Although in essence correct, the latter result is of no immediate practical use, as it treats trivial relations such as

$$D_{t_n}\tau\cdot\tau \;\equiv\; 0, \tag{88}$$

or any odd degree polynomial of D-operators acting on $\tau\cdot\tau$ as a bilinear KP equation.

Therefore, if one wants to count the number of nontrivial KP-type relations of weight n among the tau functions, one should first count the number of partitions of n into an odd number of integers, which is known to be[28]

$$\sum_{k=1}^{\infty} (-1)^{k-1}\, p(n-k^2) \;=\; p(n-1)-p(n-4)+p(n-9)-\dots\ , \tag{89}$$

and hence one has that the number of (nontrivial) bilinear KP equations of weight n is given by :

$$\#\{\mathrm{KP}_n\} = \sum_{k=2}^{\infty} (-1)^{k}\, p(n-k^2) \;=\; p(n-4)-p(n-9)+\dots \tag{90}$$

(A result conjectured in[13] but which does not appear to be widely known). For example, the number of KP_{10} equations listed in the appendix is exactly $10 = p(6) - p(1)$.

It would be interesting to know whether the natural gradation of the bilinear equations of a certain weight we obtain as a by-product of our construction, can also be described in purely combinatorial terms, i.e. : whether the exact number of equations involving a certain set of variables follows a combinatorial rule as well. A striking observation in this context is that it appears there will always be exactly one Hirota equation of type $\mathrm{KP}_n(t_{i\le 4})$, at each weight level $n \ge 4$.

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Appendix: Basic equations of weights 9 and 10

Bilinear KP-equations of weight 9, involving variables t_i up to the indicated weight :

$$\begin{aligned}\mathrm{KP}_9(t_{i\leqslant 4}) \equiv \mathcal{P}_{7x,t_2} &+6\mathcal{P}_{3x,3t_2} - 2\mathcal{P}_{4x,t_2,t_3} - 24\mathcal{P}_{3t_2,t_3} \\ &+64\mathcal{P}_{x,t_2,2t_3} - 9\mathcal{P}_{5x,t_4} - 36\mathcal{P}_{x,2t_2,t_4} = 0\end{aligned} \tag{A·1}$$

$$\begin{aligned}\mathrm{KP}_9(t_{i\leqslant 5}) \equiv \mathcal{P}_{7x,t_2} &+10\mathcal{P}_{3x,3t_2} - 10\mathcal{P}_{4x,t_2,t_3} - 40\mathcal{P}_{3t_2,t_3} - 5\mathcal{P}_{5x,t_4} \\ &+60\mathcal{P}_{x,2t_2,t_4} + 80\mathcal{P}_{2x,t_3,t_4} - 96\mathcal{P}_{2x,t_2,t_5} = 0\end{aligned} \tag{A·2}$$

$$\mathrm{KP}_9(t_{i\leqslant 6}) \equiv 3\mathcal{P}_{2x,t_3,t_4} - \mathcal{P}_{3t_2,t_3} - 2\mathcal{P}_{3x,t_6} = 0 \tag{A·3}$$

$$\begin{aligned}\widetilde{\mathrm{KP}}_9(t_{i\leqslant 6}) \equiv 4\mathcal{P}_{3x,3t_2} &-3\mathcal{P}_{4x,t_2,t_3} - 24\mathcal{P}_{x,t_2,2t_3} - 11\mathcal{P}_{3t_2,t_3} \\ &+30\mathcal{P}_{2x,t_3,t_4} + 36\mathcal{P}_{x,2t_2,t_4} - 36\mathcal{P}_{2x,t_2,t_5} \\ &-36\mathcal{P}_{t_4,t_5} + 40\mathcal{P}_{t_3,t_6} = 0\end{aligned} \tag{A·4}$$

$$\begin{aligned}\mathrm{KP}_9(t_{i\leqslant 7}) \equiv 108\mathcal{P}_{t_4,t_5} &-64\mathcal{P}_{t_3,t_6} - 72\mathcal{P}_{t_2,t_7} + 36\mathcal{P}_{x,t_2,2t_3}\\ &-27\mathcal{P}_{x,2t_2,t_4} + 30\mathcal{P}_{2x,t_3,t_4} + 36\mathcal{P}_{2x,t_2,t_5}\\ &-40\mathcal{P}_{3x,t_6} - 7\mathcal{P}_{3x,3t_2} = 0\end{aligned} \tag{A·5}$$

$$\begin{aligned}\mathrm{KP}_9(t_{i\leqslant 8}) \equiv 3\mathcal{P}_{x,t_2,2t_3} &+3\mathcal{P}_{x,2t_2,t_4} + 6\mathcal{P}_{2x,t_3,t_4} + 3\mathcal{P}_{2x,t_2,t_5} + 9\mathcal{P}_{t_4,t_5}\\ &-\mathcal{P}_{3x,t_6} + 4\mathcal{P}_{t_3,t_6} - 6\mathcal{P}_{t_2,t_7} - 21\mathcal{P}_{x,t_8} = 0\end{aligned} \tag{A·6}$$

Bilinear KP-equations of weight 10, involving variables t_i up to the indicated weight :

$$\begin{aligned}\mathrm{KP}_{10}(t_{i\leqslant 4}) \equiv 512\mathcal{P}_{x,3t_3} &-432\mathcal{P}_{3t_2,t_4} - 108\mathcal{P}_{2x,4t_2}\\ &+432\mathcal{P}_{3x,2t_2,t_3} - 96\mathcal{P}_{4x,2t_3} - 324\mathcal{P}_{4x,t_2,t_4}\\ &+27\mathcal{P}_{6x,2t_2} - 12\mathcal{P}_{7x,t_3} + \mathcal{P}_{10x} = 0\end{aligned} \tag{A·7}$$

$$\begin{aligned}\mathrm{KP}_{10}(t_{i\leqslant 5}) \equiv 128\mathcal{P}_{x,3t_3} &-108\mathcal{P}_{3t_2,t_4} + 18\mathcal{P}_{2x,4t_2}\\ &-12\mathcal{P}_{3x,2t_2,t_3} - 4\mathcal{P}_{4x,2t_3} + 9\mathcal{P}_{4x,t_2,t_4}\\ &-36\mathcal{P}_{5x,t_5} + 3\mathcal{P}_{6x,2t_2} + 2\mathcal{P}_{7x,t_3} = 0\end{aligned} \tag{A·8}$$

$$\begin{aligned}\widetilde{\mathrm{KP}}_{10}(t_{i\leqslant 5}) \equiv 240\mathcal{P}_{2t_2,2t_3} &-320\mathcal{P}_{x,3t_3} - 540\mathcal{P}_{2x,2t_4}\\ &+576\mathcal{P}_{2x,t_3,t_5} - 45\mathcal{P}_{2x,4t_2} - 80\mathcal{P}_{4x,2t_3}\\ &+180\mathcal{P}_{4x,t_2,t_4} - 15\mathcal{P}_{6x,2t_2} + 4\mathcal{P}_{7x,t_3} = 0\end{aligned} \tag{A·9}$$

$$\begin{aligned}\widehat{\mathrm{KP}}_{10}(t_{i\leqslant 5}) \equiv 20\mathcal{P}_{2t_2,2t_3} &-60\mathcal{P}_{x,t_2,t_3,t_4} + 36\mathcal{P}_{x,2t_2,t_5}\\ &-10\mathcal{P}_{3x,2t_2,t_3} + 15\mathcal{P}_{4x,t_2,t_4} - \mathcal{P}_{6x,2t_2} = 0\end{aligned} \tag{A·10}$$

$$\begin{aligned}\mathrm{KP}_{10}(t_{i\leqslant 6}) \equiv 4\mathcal{P}_{2t_2,2t_3} &+4\mathcal{P}_{x,t_2,t_3,t_4} - 12\mathcal{P}_{2x,2t_4}\\ &-12\mathcal{P}_{x,2t_2,t_5} + 16\mathcal{P}_{2x,t_2,t_6} - \mathcal{P}_{2x,4t_2}\\ &+2\mathcal{P}_{3x,2t_2,t_3} - \mathcal{P}_{4x,t_2,t_4} = 0\end{aligned} \tag{A·11}$$

$$\begin{aligned}\widetilde{\mathrm{KP}}_{10}(t_{i\leqslant 6}) \equiv 288\mathcal{P}_{2t_5} &-300\mathcal{P}_{t_4,t_6} + 25\mathcal{P}_{2t_2,2t_3} - 4\mathcal{P}_{x,3t_3} - 6\mathcal{P}_{3t_2,t_4}\\ &-9\mathcal{P}_{2x,4t_2} + \mathcal{P}_{3x,2t_2,t_3} - 13\mathcal{P}_{4x,2t_3} + 18\mathcal{P}_{4x,t_2,t_4}\\ &+\mathcal{P}_{6x,2t_2} - \mathcal{P}_{7x,t_3} = 0\end{aligned} \tag{A·12}$$

$$\begin{aligned}\mathrm{KP}_{10}(t_{i\leqslant 7}) \equiv 2\mathcal{P}_{2t_2,2t_3} &+\mathcal{P}_{3t_2,t_4} - 2\mathcal{P}_{x,t_2,t_3,t_4} - 3\mathcal{P}_{2x,2t_4}\\ &-4\mathcal{P}_{2x,t_3,t_5} + 2\mathcal{P}_{2x,t_2,t_6} + 4\mathcal{P}_{3x,t_7} = 0\end{aligned} \tag{A·13}$$

$$\widetilde{\mathrm{KP}}_{10}(t_{i\leqslant 7}) \equiv 252\mathcal{P}_{t_4,t_6} - 288\mathcal{P}_{t_3,t_7} + 21\mathcal{P}_{2t_2,2t_3} + 28\mathcal{P}_{x,3t_3} - 21\mathcal{P}_{3x,2t_2,t_3} + 7\mathcal{P}_{4x,2t_3} + \mathcal{P}_{7x,t_3} = 0 \tag{A·14}$$

$$\mathrm{KP}_{10}(t_{i\leqslant 8}) \equiv 8\mathcal{P}_{t_4,t_6} - 12\mathcal{P}_{t_2,t_8} + \mathcal{P}_{3t_2,t_4} + 4\mathcal{P}_{x,t_2,t_3,t_4} - \mathcal{P}_{4x,t_2,t_4} = 0 \tag{A·15}$$

$$\begin{aligned}\mathrm{KP}_{10}(t_{i\leqslant 9}) \equiv\ & 12\mathcal{P}_{2t_5} + 19\mathcal{P}_{t_4,t_6} + 4\mathcal{P}_{t_3,t_7} - 21\mathcal{P}_{t_2,t_8} - 3\mathcal{P}_{2t_2,2t_3} \\ & + 2\mathcal{P}_{x,3t_3} - \mathcal{P}_{3t_2,t_4} + 18\mathcal{P}_{x,t_2,t_3,t_4} \\ & + 12\mathcal{P}_{2x,2t_4} + 6\mathcal{P}_{x,2t_2,t_5} + 18\mathcal{P}_{2x,t_3,t_5} \\ & - 10\mathcal{P}_{3x,t_7} - 56\mathcal{P}_{x,t_9} = 0\end{aligned} \tag{A·16}$$

Bilinear mKP-equations of weight 9, involving variables t_i up to the indicated weight :

$$\begin{aligned}\mathrm{mKP}_9(t_{i\leqslant 3}) \equiv\ & 512\mathcal{Y}_{3t_3} + 576\mathcal{Y}_{3t_2,t_3} - 1152\mathcal{Y}_{x,t_2,2t_3} + 216\mathcal{Y}_{x,4t_2} \\ & + 432\mathcal{Y}_{2x,2t_2,t_3} - 384\mathcal{Y}_{3x,2t_3} - 36\mathcal{Y}_{3x,3t_2} - 288\mathcal{Y}_{4x,t_2,t_3} \\ & + 54\mathcal{Y}_{5x,2t_2} + 60\mathcal{Y}_{6x,t_3} + 9\mathcal{Y}_{7x,t_2} + \mathcal{Y}_{9x} = 0\end{aligned} \tag{A·17}$$

$$\begin{aligned}\mathrm{mKP}_9(t_{i\leqslant 4}) \equiv\ & 48\mathcal{Y}_{3t_2,t_3} - 128\mathcal{Y}_{x,t_2,2t_3} + 72\mathcal{Y}_{x,2t_2,t_4} - 36\mathcal{Y}_{x,4t_2} \\ & + 48\mathcal{Y}_{2x,2t_2,t_3} - 12\mathcal{Y}_{3x,3t_2} + 28\mathcal{Y}_{4x,t_2,t_3} - 18\mathcal{Y}_{5x,t_4} \\ & - 3\mathcal{Y}_{5x,2t_2} + \mathcal{Y}_{7x,t_2} = 0\end{aligned} \tag{A·18}$$

$$\begin{aligned}\widetilde{\mathrm{mKP}}_9(t_{i\leqslant 4}) \equiv\ & 72\mathcal{Y}_{t_2,t_3,t_4} + 12\mathcal{Y}_{3t_2,t_3} + 16\mathcal{Y}_{x,t_2,2t_3} - 36\mathcal{Y}_{x,2t_2,t_4} \\ & - 72\mathcal{Y}_{2x,t_3,t_4} + 18\mathcal{Y}_{x,4t_2} + 12\mathcal{Y}_{2x,2t_2,t_3} - 18\mathcal{Y}_{3x,t_2,t_4} \\ & - 3\mathcal{Y}_{3x,3t_2} - 8\mathcal{Y}_{4x,t_2,t_3} + 6\mathcal{Y}_{5x,2t_2} + \mathcal{Y}_{7x,t_2} = 0\end{aligned} \tag{A·19}$$

$$\begin{aligned}\widehat{\mathrm{mKP}}_9(t_{i\leqslant 4}) \equiv\ & 216\mathcal{Y}_{x,4t_2} - 128\mathcal{Y}_{3x,2t_3} + 144\mathcal{Y}_{3x,t_2,t_4} + 36\mathcal{Y}_{3x,3t_2} \\ & - 144\mathcal{Y}_{2x,2t_2,t_3} - 240\mathcal{Y}_{4x,t_2,t_3} + 72\mathcal{Y}_{5x,t_4} + 18\mathcal{Y}_{5x,2t_2} \\ & + 28\mathcal{Y}_{6x,t_3} - 3\mathcal{Y}_{7x,t_2} + \mathcal{Y}_{9x} = 0\end{aligned} \tag{A·20}$$

$$\begin{aligned}\mathrm{mKP}_9(t_{i\leqslant 5}) \equiv\ & -120\mathcal{Y}_{t_2,t_3,t_4} + 120\mathcal{Y}_{x,2t_4} - 128\mathcal{Y}_{x,t_3,t_5} + 20\mathcal{Y}_{2x,t_3,t_4} \\ & + 120\mathcal{Y}_{2x,t_2,t_5} + 30\mathcal{Y}_{x,4t_2} - 30\mathcal{Y}_{2x,2t_2,t_3} + 20\mathcal{Y}_{4x,t_5} \\ & - 45\mathcal{Y}_{4x,t_2,t_3} + 10\mathcal{Y}_{5x,t_4} + 3\mathcal{Y}_{6x,t_3} = 0\end{aligned} \tag{A·21}$$

$$\begin{aligned}\widetilde{\mathrm{mKP}}_9(t_{i\leqslant 5}) \equiv\ & 80\mathcal{Y}_{t_2,t_3,t_4} - 48\mathcal{Y}_{2t_2,t_5} - 30\mathcal{Y}_{x,2t_2,t_4} + 15\mathcal{Y}_{x,4t_2} \\ & - 20\mathcal{Y}_{3x,t_2,t_4} + 3\mathcal{Y}_{5x,2t_2} = 0\end{aligned} \tag{A·22}$$

$$\widehat{\mathrm{mKP}}_9(t_{i\leqslant 5}) \equiv 160\mathcal{Y}_{t_2,t_3,t_4} - 80\mathcal{Y}_{2x,t_3,t_4} - 96\mathcal{Y}_{2x,t_2,t_5} + 40\mathcal{Y}_{2x,2t_2,t_3} - 40\mathcal{Y}_{3x,t_2,t_4} + 20\mathcal{Y}_{4x,t_2,t_3} - 10\mathcal{Y}_{5x,t_4} + 5\mathcal{Y}_{5x,2t_2} + \mathcal{Y}_{7x,t_2} = 0 \tag{A·23}$$

$$\overline{\mathrm{mKP}}_9(t_{i\leqslant 5}) \equiv 144\mathcal{Y}_{4x,t_5} + 90\mathcal{Y}_{5x,t_4} + 45\mathcal{Y}_{5x,2t_2} + 40\mathcal{Y}_{6x,t_3} + 360\mathcal{Y}_{x,4t_2} - 320\mathcal{Y}_{3x,2t_3} - 360\mathcal{Y}_{4x,t_2,t_3} + \mathcal{Y}_{9x} = 0 \tag{A·24}$$

$$\mathrm{mKP}_9(t_{i\leqslant 6}) \equiv 576\mathcal{Y}_{t_4,t_5} - 640\mathcal{Y}_{t_3,t_6} + 20\mathcal{Y}_{3t_2,t_3} + 90\mathcal{Y}_{x,2t_2,t_4} + 15\mathcal{Y}_{x,4t_2} - 80\mathcal{Y}_{3x,2t_3} + 60\mathcal{Y}_{4x,t_5} - 10\mathcal{Y}_{3x,3t_2} - 45\mathcal{Y}_{4x,t_2,t_3} - 6\mathcal{Y}_{5x,t_4} + 15\mathcal{Y}_{5x,2t_2} + 5\mathcal{Y}_{6x,t_3} = 0 \tag{A·25}$$

$$\widetilde{\mathrm{mKP}}_9(t_{i\leqslant 6}) \equiv -40\mathcal{Y}_{t_2,t_3,t_4} + 60\mathcal{Y}_{x,2t_4} - 128\mathcal{Y}_{x,t_3,t_5} + 80\mathcal{Y}_{x,t_2,t_6} + 30\mathcal{Y}_{x,2t_2,t_4} + 10\mathcal{Y}_{x,4t_2} - 20\mathcal{Y}_{3x,t_2,t_4} + 20\mathcal{Y}_{4x,t_5} - 15\mathcal{Y}_{4x,t_2,t_3} + 3\mathcal{Y}_{6x,t_3} = 0 \tag{A·26}$$

$$\widehat{\mathrm{mKP}}_9(t_{i\leqslant 6}) \equiv 96\mathcal{Y}_{2t_2,t_5} - 40\mathcal{Y}_{2x,t_3,t_4} - 80\mathcal{Y}_{3x,t_6} - 20\mathcal{Y}_{2x,2t_2,t_3} + 30\mathcal{Y}_{3x,t_2,t_4} + 5\mathcal{Y}_{3x,3t_2} + 10\mathcal{Y}_{5x,t_4} - \mathcal{Y}_{5x,2t_2} = 0 \tag{A·27}$$

$$\mathrm{mKP}_9(t_{i\leqslant 7}) \equiv 54\mathcal{Y}_{t_4,t_5} - 60\mathcal{Y}_{t_3,t_6} + 6\mathcal{Y}_{x,2t_4} - 15\mathcal{Y}_{2t_2,t_5} - 120\mathcal{Y}_{x,t_2,t_6} + 96\mathcal{Y}_{2x,t_7} + 16\mathcal{Y}_{3t_2,t_3} - 63\mathcal{Y}_{x,t_2,2t_3} + 27\mathcal{Y}_{x,2t_2,t_4} + 42\mathcal{Y}_{2x,t_3,t_4} + 63\mathcal{Y}_{2x,t_2,t_5} - 25\mathcal{Y}_{3x,t_6} + 3\mathcal{Y}_{x,4t_2} - 24\mathcal{Y}_{3x,2t_3} = 0 \tag{A·28}$$

$$\widetilde{\mathrm{mKP}}_9(t_{i\leqslant 7}) \equiv 28\mathcal{Y}_{t_3,t_6} - 36\mathcal{Y}_{t_2,t_7} - 10\mathcal{Y}_{3t_2,t_3} + 27\mathcal{Y}_{x,t_2,2t_3} - 18\mathcal{Y}_{2x,t_2,t_5} + 17\mathcal{Y}_{3x,t_6} - 9\mathcal{Y}_{3x,t_2,t_4} + \mathcal{Y}_{3x,3t_2} = 0 \tag{A·29}$$

$$\mathrm{mKP}_9(t_{i\leqslant 8}) \equiv 280\mathcal{Y}_{t_3,t_6} - 1920\mathcal{Y}_{t_2,t_7} + 2730\mathcal{Y}_{x,t_8} - 840\mathcal{Y}_{t_2,t_3,t_4} + 105\mathcal{Y}_{x,2t_4} - 1050\mathcal{Y}_{x,t_2,t_6} + 336\mathcal{Y}_{2x,t_2,t_5} + 490\mathcal{Y}_{3x,t_6} - 105\mathcal{Y}_{x,4t_2} - 35\mathcal{Y}_{3x,3t_2} + 9\mathcal{Y}_{7x,t_2} = 0 \tag{A·30}$$

$$\begin{aligned}\mathrm{mKP}_9(t_{i\leqslant 9}) \equiv\, & 4\mathcal{Y}_{t_9} + 6\mathcal{Y}_{t_4,t_5} + 2\mathcal{Y}_{t_3,t_6} - 9\mathcal{Y}_{t_2,t_7} - 6\mathcal{Y}_{x,t_8} \\ & +2\mathcal{Y}_{3t_3} + 9\mathcal{Y}_{t_2,t_3,t_4} + 9\mathcal{Y}_{x,2t_4} + 12\mathcal{Y}_{x,t_3,t_5} - 9\mathcal{Y}_{x,t_2,t_6} \\ & -15\mathcal{Y}_{2x,t_7} - 2\mathcal{Y}_{3t_2,t_3} - 3\mathcal{Y}_{x,t_2,2t_3} - 3\mathcal{Y}_{x,2t_2,t_4} \\ & +3\mathcal{Y}_{2x,t_3,t_4} = 0\end{aligned} \tag{A·31}$$

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