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On a direct procedure for the disclosure of Lax pairs and Bäcklund transformations ☆

Franklin Lambert *, Johan Springael

Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

Abstract

A direct and unifying scheme for the disclosure of bilinear Bäcklund transformations and linear Lax systems associated with soliton equations is presented. The scheme is based on a concept of scale invariance and on the use of a class of partitional polynomials: the binary Bell polynomials. The applicability of the procedure is tested on a variety of soliton equations.

1. Introduction

Among the direct algebraic methods applicable to nonlinear partial differential equations (NLPDEs) for the investigation of their integrability there is one which has proved particularly powerful: it is the bilinear method developed by Hirota [1,2]. Not only have the Hirota representations of soliton systems produced most of the known families of closed form solutions, they have also lead to the disclosure of major integrability features of these systems, such as (auto) Bäcklund transformations (BTs), associated linear problems (Lax pairs) and infinite sets of conserved quantities [3,4]. A key element has often been the disclosure of a bilinear BT acting at the level of a suitable Hirota representation of the NLPDE. These bilinear BTs consist of a set of parameter-dependent bilinear equations for a number of pairs of functions (G_i, G_i') – one for each Hirota function G_i appearing in the representation – which are compatible if the G_i's satisfy the equations of that representation, and which can be transformed into a corresponding linear system of the Lax-AKNS type.

Bilinear BTs have been derived for a variety of soliton equations by applying well chosen "exchange formulas" to a particular ansatz related to a Hirota representation of the NLPDE [3,5]. Yet, from the practical point of view, these derivations are not as direct as one would wish. Given a soliton equation one is still confronted with the following problem: "How should one proceed in order to obtain the associated (bi)linear systems without relying on clever guesswork?" In the case of sech squared soliton systems derivable from a quadratic Hirota equation for one function G and admitting a bilinear BT in terms of one Hirota pair (G, G') it was found [6] that a direct and unifying procedure could be developed by reformulating the problem in terms of two "mixing" variables $v = \ln(G'/G)$ and $w = \ln(G'G)$. Starting from a "primary" version of the original NLPDE, which can be derived from that equation by an argument of scale invariance, the procedure makes use of a basis of partitional polynomials – the binary Bell polynomials [7] – which are closely related to Hirota's standard bilinear expressions.

A remaining problem was to extend the "mixing variable procedure" to the case in which the bilinear BT involves more than one Hirota pair. A difficulty to be overcome in this case is the fact that the number of mixing variables exceeds the number of Hirota functions. In this paper we resolve this difficulty and show that the procedure can be adapted so as to benefit from this excess. We apply it to three examples for which bilinear BTs, involving four Hirota functions, and corresponding Lax pairs, are obtained in a unified fashion: the sine-Gordon equation, the modified KdV

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* *E-mail address:* tenasecr@vub.ac.be (F. Lambert).

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equation and the coupled AKNS system. In order to provide a complete picture of the method we start our discussion with the case of NLPDEs which can be derived from a quadratic Hirota equation, considering two simple examples: the KdV equation and the Sawada–Kotera equation.

2. Bilinear BTs for NLPDEs derivable from a quadratic Hirota equation

The simplest soliton equations, from the bilinear point of view, are 1 + 1-dimensional NLPDEs which can be derived (through differentiation) from one Hirota equation of the *quadratic* type

$$\mathscr{F}(G) = \sum_i c_i D_x^{m_i} D_t^{n_i} G \cdot G = 0, \tag{1}$$

where $m_i,\ n_i =$ integer or zero, $m_i + n_i =$ even, $c_i =$ constant, and where the D-operators are defined by their action on an ordered pair of functions $F(x,t)$ and $G(x,t)$ by the rule

$$D_x^m D_t^n F \cdot G = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n F(x,t) G(x',t')\big|_{x'=x,\, t'=t.} \tag{2}$$

Such NLPDEs are said to admit a two-field bilinear BT if there exists a system of two bilinear equations for one Hirota pair (G, G') ($p_j, q_j, r_j, s_j =$ integer or zero, $c_{ij} =$ constant)

$$\sum_j c_{1j} D_x^{p_j} D_t^{q_j} G' \cdot G = 0, \quad \sum_j c_{2j} D_x^{r_j} D_t^{s_j} G' \cdot G = 0, \tag{3}$$

which are compatible if G satisfies Eq. (1) and which imply the relation

$$G^2 \mathscr{F}(G') - G'^2 \mathscr{F}(G) = 0. \tag{4}$$

The problem that we consider is that of finding such a system (if it exists) for a given $\mathscr{F}(G)$. A current but quite tricky procedure is to start from Eq. (4) and to try to decompose it into a pair of Eqs. (3) by using appropriate "exchange formulas" [3,5].

A more straightforward method is to introduce the mixing variables:

$$v = \ln(G'/G) \quad \text{and} \quad w = \ln(G'G) \tag{5}$$

in terms of which each standard Hirota term is mapped onto a binary Bell polynomial according to the identity [7]

$$\mathscr{Y}_{px,qt}(v,w) \equiv (G'G)^{-1} D_x^p D_t^q G' \cdot G\Big|_{G=\exp((w-v)/2),\, G'=\exp((w+v)/2).} \tag{6}$$

The $\mathscr{Y}$-polynomials involve even-order derivatives of w and odd-order derivatives of v. They are defined in terms of the exponential Bell polynomials [8]

$$Y_{mx,nt}(f) \equiv e^{-f} \partial_x^m \partial_t^n e^f \tag{7}$$

as follows:

$$\mathscr{Y}_{mx,nt}(v,w) \equiv Y_{mx,nt}(f)\Big|_{f_{px,qt} = \begin{cases} v_{px,qt} & \text{if } p+q = \text{odd}, \\ w_{px,qt} & \text{if } p+q = \text{even} \end{cases}} \tag{8}$$

with the understanding that $f_{px,qt} \equiv \partial_x^p \partial_t^q f$.

Each polynomial $\mathscr{Y}_{mx,nt(v,w)}$, constructed with the derivatives of dimensionless variables v and w, is a homogeneous expression of weight $p + qr$ if r stands for the dimension of t and the dimension of x is chosen equal to 1. Three points are in favour of the introduction of the alternative (v,w) to the Hirota pair (G',G). The first one is that the $\mathscr{Y}$-polynomials inherit the striking partitional structure of the exponential Bell polynomials (combined with a simple rule which assigns labels v and w according to the parity of the number of derivatives: odd or even):

$$\mathscr{Y}_x(v) = v_x, \quad \mathscr{Y}_{2x}(v,w) = w_{2x} + v_x^2, \quad \mathscr{Y}_{x,t}(v,w) = w_{xt} + v_x v_t, \quad \mathscr{Y}_{3x}(v,w) = v_{3x} + 3v_x w_{2x} + v_x^3, \ldots \tag{9}$$

In the particular case $F = G$ one has

$$G^{-2} D_x^m D_t^n G \cdot G \equiv \mathscr{Y}_{mx,nt}(0, Q = 2\ln G) \equiv \begin{cases} 0 & \text{if } m+n = \text{odd}, \\ P_{mx,nt}(Q) & \text{if } m+n = \text{even} \end{cases} \tag{10}$$

with

$$P_{2x}(Q) = Q_{2x}, \quad P_{x,t}(Q) = Q_{xt}, \quad P_{4x}(Q) = Q_{4x} + 3Q_{2x}^2, \quad P_{6x}(Q) = Q_{6x} + 15Q_{2x}Q_{4x} + 15Q_{2x}^3, \ldots \tag{11}$$

Hence it is clear that bilinear equations of form (3) correspond, through map (5), to linear combinations of easily recognizable $\mathscr{Y}$-polynomials, whereas the quadratic equation (1) is mapped onto an equation involving a linear combination of equally recognizable "even part" P-polynomials.

The second point is the logarithmic linearizability of the $\mathscr{Y}$-polynomials [7]

$$\mathscr{Y}_{px,qt}(v = \ln\psi, w = Q + \ln\psi) = \psi^{-1}L_{p,q}(Q)\psi \tag{12}$$

with

$$L_{p,q}(Q) = \sum_{j=0}^{p}\sum_{k=0}^{q}\binom{p}{j}\binom{q}{k}\mathscr{Y}_{jx,kt}(0,Q)\partial_x^{p-j}\partial_t^{q-k} \tag{13}$$

indicating that each bilinear equation (3) is mapped by the transformation:

$$w = v + Q, \quad v = \ln\psi \tag{14}$$

onto a linear partial differential equation for $\psi = G'/G$, with coefficients depending on the even-order derivatives of $Q = 2\ln G$.

The third point is the fact that transformation (5), which maps the quadratic Hirota equation (2) onto the NLPDE

$$E(Q) \equiv \sum_i c_i P_{m_i x, n_i t}(Q) = 0 \tag{15}$$

maps condition (4) onto

$$C(v,w) \equiv E(Q' = w + v) - E(Q = w - v) = 0. \tag{16}$$

This two-field condition can be regarded as the natural ansatz for a bilinear BT: it is the homogeneous constraint between the "primary field" Q and a replica Q' in the form of a linear combination

$$\alpha E(Q') + \beta E(Q) = 0, \tag{17}$$

which satisfies the requirement $\alpha + \beta = 0$ so that the condition can hold independently of the primary equation (15) when $Q' = Q + \epsilon\phi$ with $\epsilon =$ small parameter. We call it the Hirota ansatz on account of its equivalence with relation (4). Thus, it is seen that the introduction of the variables v and w enables us to reformulate our problem as follows.

Given an NLPDE for a "primary" field Q of form (15), find a pair of constraints:

$$\sum_j c_{1j}\mathscr{Y}_{p_j x, q_j t}(v,w) = 0, \quad \sum_j c_{2j}\mathscr{Y}_{r_j x, s_j t}(v,w) = 0, \tag{18}$$

which are compatible if $Q = w - v$ satisfies Eq. (15) and which imply relation (16).

We shall see that the search for such constraints can be performed systematically (if not algorithmically) through direct bilinearization of the two-field condition (16), i.e., by expressing $C(v,w)$ in terms of $\mathscr{Y}$-polynomials (and their derivatives) and by decoupling the resulting condition into a pair of constraints (18), through the introduction of *one* such constraint of lowest possible weight. It is clear from the above formulas (6) and (12) and (13) that every $\mathscr{Y}$-system (18) into which Eq. (16) can be decoupled, produces a bilinear representation (3) of that equation in terms of the Hirota pair $G = \exp((w-v)/2)$, $G' = \exp((w+v)/2)$, as well as an equivalent linear system for $\psi = \mathrm{e}^v$

$$\mathscr{L}_1(Q)\psi = \sum_j c_{1j}L_{p_j,q_j}(Q)\psi = 0, \quad \mathscr{L}_2(Q)\psi = \sum_j c_{2j}L_{r_j,s_j}(Q)\psi = 0. \tag{19}$$

Thus, having found such a $\mathscr{Y}$-system, it is a straightforward matter to check the Bäcklund property of the corresponding bilinear system (3) by verifying that the compatibility of the linear system (19) is subject to a condition on Q which is satisfied as a result of Eq. (15). If this is the case, Eqs. (18) produce a coupled system of NLPDEs for $Q = w - v$ and $Q' = w + v$ with the basic Bäcklund property. The actual BT will then be given by a corresponding system of NLPDEs for the potential fields $V = Q_x$ and $V' = Q'_x$. This system should contain an arbitrary constant which plays the role of a Bäcklund parameter. The $\mathscr{Y}$-system can usually be chosen so as to contain several arbitrary parameters (with weight $\neq 0$) which appear either as arbitrary coefficients c_{ij}, or as integration constants. Redundant parameters (to be set equal to zero) can be identified at the level of the linear system (19) through application of an appropriate gauge transformation on ψ.

Example 1. The simplest example is that of the KdV equation

$$u_t + u_{3x} + uu_x = 0, \tag{20}$$

which is invariant under the scale transformation

$$x \to \lambda x, \quad t \to \lambda^3 t, \quad u \to \lambda^{-2} u, \tag{21}$$

indicating that u has dimension -2. A dimensionless field Q can be introduced by setting $u = cQ_{2x}$, with $c =$ dimensionless parameter to be determined. The resulting equation for Q leads (after one integration with respect to x with zero boundary condition) to a primary (potential) equation

$$Q_{xt} + Q_{4x} + \frac{c}{2} Q_{2x}^2 = 0, \tag{22}$$

which can be cast into the form

$$E(Q) \equiv Q_{xt} + Q_{4x} + 3Q_{2x}^2 \equiv P_{xt}(Q) + P_{4x}(Q) \equiv G^{-2}(D_x D_t + D_x^4) G \cdot G\big|_{G=\mathrm{e}^{Q/2}} = 0 \tag{23}$$

if one chooses $c = 6$.

The two-field condition (16) for KdV is

$$\begin{aligned} C(v,w) &\equiv P_{xt}(w+v) - P_{xt}(w-v) + P_{4x}(w+v) - P_{4x}(w-v) \\ &\equiv 2(v_{xt} + v_{4x} + 6v_{2x}w_{2x}) \\ &\equiv 2\partial_x[\mathscr{Y}_t(v) + \mathscr{Y}_{3x}(v,w)] + 6R(v,w) = 0 \end{aligned} \tag{24}$$

with

$$R(v,w) = \mathrm{Wronskian}\,[\mathscr{Y}_{2x}(v,w), \mathscr{Y}_x(v)]. \tag{25}$$

In order to decouple this condition into a pair of constraints (18) one imposes one such constraint (of lowest possible weight) which enables one to express $R(v,w)$ as the x-derivative of a linear combination of $\mathscr{Y}$-polynomials. The simplest possible choice is the constraint of weight 2

$$\mathscr{Y}_{2x}(v,w) \equiv w_{2x} + v_x^2 = 0, \tag{26}$$

which implies that $R(v,w) = 0$. In order to obtain a decomposition which contains an arbitrary parameter we consider the alternative

$$\mathscr{Y}_{2x}(v,w) = \lambda, \quad \lambda = \text{arbitrary parameter of weight } 2, \tag{27}$$

on account of which $R(v,w) = \lambda v_{2x} = \lambda \partial_x \mathscr{Y}_x(v)$.

Thus, it is seen that the two-field condition (24) can be decoupled into the following parameter-dependent $\mathscr{Y}$-system:

$$\mathscr{Y}_{2x}(v,w) - \lambda = 0, \quad \mathscr{Y}_t(v) + \mathscr{Y}_{3x}(v,w) + 3\lambda \mathscr{Y}_x(v) = 0. \tag{28}$$

Its compatibility is subject to that of the corresponding system for ψ (setting $w = v + Q$ and $v = \ln \psi$):

$$\psi_{2x} + (Q_{2x} - \lambda)\psi = 0, \quad \psi_t + \psi_{3x} + 3(Q_{2x} + \lambda)\psi_x = 0, \tag{29}$$

i.e., to the (λ-independent) condition which is found after the elimination of ψ

$$(Q_{xt} + Q_{4x} + 3Q_{2x})_x \equiv \partial_x E(Q) = 0. \tag{30}$$

It follows that Eq. (30) can be regarded as the integrability condition for system (29) or the equivalent system:

$$\psi_{2x} + (Q_{2x} - \lambda)\psi = 0, \quad \psi_t + (2Q_{2x} + 4\lambda)\psi_x - Q_{3x}\psi = 0, \tag{31}$$

which is the well-known Lax system associated with KdV.

The bilinear equivalent of system (28) for $G = \exp((w-v)/2)$ and $G' = \exp((w+v)/2)$

$$D_x^2 G' \cdot G = \lambda G' G, \quad (D_t + D_x^3 + 3\lambda D_x) G' \cdot G = 0 \tag{32}$$

is the bilinear BT for KdV proposed by Hirota [3].

It may be noticed that the two-field condition (24) could also have been decoupled into the slightly more general $\mathscr{Y}$-system ($\lambda, \nu, \mu =$ arbitrary parameters):

$$\mathscr{Y}_{2x}(v,w) + \mu \mathscr{Y}_x(v) - \lambda = 0, \quad \mathscr{Y}_t(v) + \mathscr{Y}_{3x}(v,w) + 3\lambda \mathscr{Y}_x(v) - \nu = 0. \tag{33}$$

The redundancy of the parameters μ and v is easily established at the level of the equivalent system for $\psi = e^v$ (setting $w = v + Q$):

$$\psi_{2x} + \mu\psi_x + (Q_{2x} - \lambda)\psi = 0, \quad \psi_t + (2Q_{2x} + 4\lambda + \mu^2)\psi_x + (\mu Q_{2x} - Q_{3x} - \nu - \lambda\mu)\psi = 0 \tag{34}$$

by noticing that this system can be transformed into a system of form (31) by application of the gauge transformation:

$$\psi \to \tilde{\psi} = \psi e^{-(\alpha x + \beta t)} \tag{35}$$

with $\alpha = -\mu/2,\ \beta = \mu^3/2 + 3\lambda\mu + \nu$.

Example 2. We consider the Sawada–Kotera equation

$$u_t + u_{5x} + 15uu_{3x} + 15u_x u_{2x} + 45u^2 u_x = 0. \tag{36}$$

Introducing again a dimensionless field Q by setting $u = Q_{2x}$ we are led to the primary (potential) equation

$$E(Q) \equiv Q_{xt} + Q_{6x} + 15Q_{2x}Q_{4x} + 15Q_{2x}^3 \equiv P_{xt}(Q) + P_{6x}(Q) = 0. \tag{37}$$

The corresponding two-field condition is

$$C(v,w) \equiv E(w+v) - E(w-v) \equiv 2\partial_x[\mathscr{Y}_t(v) + \mathscr{Y}_{5x}(v,w)] + 10R(v,w) = 0 \tag{38}$$

with

$$R(v,w) = -v_x w_{5x} + 2v_{2x}w_{4x} - 2v_{3x}w_{3x} + w_{2x}v_{4x} - 2v_x^2 v_{4x} - 4v_x v_{2x} v_{3x} + 6v_{2x}w_{2x}^2 + 3v_{2x}^3 - 6v_x w_{2x} w_{3x} - 2v_x^3 w_{3x} - 6v_x^2 v_{2x} w_{2x} - v_x^4 v_{2x}. \tag{39}$$

It is easy to see that the elimination of w_{2x} by means of a second-order constraint of form (27) does *not* reduce the remainder $R(v,w)$ to an expression which can be written as the x-derivative of a linear combination of $\mathscr{Y}$-polynomials. However, it is found that a third-order constraint

$$\mathscr{Y}_{3x}(v,w) \equiv v_{3x} + 3v_x w_{2x} + v_x^3 = \lambda \tag{40}$$

can be used to express $R(v,w)$ as follows:

$$R(v,w) = -\frac{1}{2}\partial_x[\mathscr{Y}_{5x}(v,w) + 3\lambda\mathscr{Y}_{2x}(v,w)], \tag{41}$$

where we have used

$$\mathscr{Y}_{5x}(v,w) = v_{5x} + 5v_x w_{4x} + 10v_{3x}w_{2x} + 10v_x^2 v_{3x} + 15v_x w_{2x}^2 + 10v_x^3 w_{2x} + v_x^5. \tag{42}$$

Thus, it is seen that the two-field condition (38) can be decoupled into the $\mathscr{Y}$-system:

$$\mathscr{Y}_{3x}(v,w) - \lambda = 0, \quad \mathscr{Y}_t(v) - \frac{3}{2}\mathscr{Y}_{5x}(v,w) - \frac{15}{2}\lambda\mathscr{Y}_{2x}(v,w) = 0. \tag{43}$$

Its compatibility is subject to that of the corresponding ψ-system ($w = v + Q,\ v = \ln\psi$):

$$\psi_{3x} + 3Q_{2x}\psi_x - \lambda\psi = 0, \quad \psi_t - \frac{3}{2}\psi_{5x} - 15Q_{2x}\psi_{3x} - \frac{15}{2}P_{4x}(Q)\psi_x - \frac{15}{2}\lambda(\psi_{2x} + Q_{2x}\psi) = 0, \tag{44}$$

i.e., to the condition

$$\left(Q_{xt} + Q_{6x} + 15Q_{2x}Q_{4x} + 15Q_{2x}^3\right)_x \equiv \partial_x E(Q) = 0. \tag{45}$$

The equivalent system:

$$\psi_{3x} + 3Q_{2x}\psi_x - \lambda\psi = 0, \quad \psi_t + 9(Q_{3x} - \lambda)\psi_{2x} - 3(Q_{4x} - 3Q_{2x}^2)\psi_x - 18\lambda Q_{2x}\psi = 0 \tag{46}$$

is the Lax system for the Sawada–Kotera equation found by Satsuma et al. [9].

The bilinear equivalent of system (43):

$$(D_x^3 - \lambda)G' \cdot G = 0, \quad \left(D_t - \frac{3}{2}D_x^5 - \frac{15}{2}\lambda D_x^2\right)G' \cdot G = 0 \tag{47}$$

is the bilinear BT for Sawada–Kotera reported in [9].

3. Bilinear BTs for NLPDEs derivable from a two-field Hirota system

It is well known that the process of bilinearization of a given NLPDE – i.e., the search for a primary system in Hirota form from which the NLPDE can be derived through differentiation – usually requires the introduction of one or more auxiliary dependent variables. Let us therefore specialize to 1 + 1-dimensional NLPDEs which can be derived from a Hirota system for two functions F and G, involving linear combinations of the following terms ($m, n, p, q, r, s =$ integer or zero, $p + q =$ even, $r + s =$ even):

$$D_x^m D_t^n F \cdot G, \quad D_x^p D_t^q G \cdot G, \quad D_x^r D_t^s F \cdot F. \tag{48}$$

These terms are mapped by the transformation:

$$V = \ln(F/G), \quad W = \ln(FG) \tag{49}$$

onto e^W times the following co-bilinearizable terms:

$$\mathcal{Y}_{mx,nt}(V, W), \quad e^{-V} P_{px,qt}(W - V), \quad e^V P_{rx,st}(W + V). \tag{50}$$

Hence, we shall concentrate on NLPDEs derivable from a primary equation

$$E(V) = 0, \tag{51}$$

which can be decomposed into a system of two equations, for the primary field V and an auxiliary field W

$$\mathcal{E}_i(V, W) = 0, \quad i = 1, 2, \tag{52}$$

involving only linear combinations of the basic terms (50).

Such NLPDEs admit a *four-field bilinear* BT if there exists a set of four bilinear equations, for two Hirota pairs $\{(F, G), (F', G')\}$, of the form ($k_i, \ell_i, m_i, n_i, p_i, q_i, r_i, s_i =$ integer or zero, $a_i, b_i, c_i, d_i =$ constant)

$$\sum_i \left[a_i D_x^{k_i} D_t^{\ell_i} G' \cdot G + b_i D_x^{m_i} D_t^{n_i} F' \cdot F + c_i D_x^{p_i} D_t^{q_i} F' \cdot G + d_i D_x^{r_i} D_t^{s_i} G' \cdot F\right] = 0, \tag{53}$$

which are compatible if F and G satisfy the bilinear representation of the NLPDE and which imply the relation

$$E(V') - E(V) = 0 \tag{54}$$

with $V' = \ln(F'/G')$.

In order to see whether such a set can be found for a given $E(V)$, one may introduce the mixing variables:

$$v_1 = \ln\frac{G'}{G}, \quad v_2 = \ln\frac{F'}{F}, \quad v_3 = \ln\frac{F'}{G}, \quad v_4 = \ln\frac{G'}{F}, \quad w_1 = \ln GG', \quad w_2 = \ln FF', \quad w_3 = \ln GF', \quad w_4 = \ln FG', \tag{55}$$

and look for a corresponding set of equations

$$\sum_i \left[a_i \mathcal{Y}_{k_i x, \ell_i t}(v_1, w_1) + b_i e^{v_3 - v_4} \mathcal{Y}_{m_i x, n_i t}(v_2, w_2) + c_i e^{v_3 - v_1} \mathcal{Y}_{p_i x, q_i t}(v_3, w_3) + d_i e^{v_3 - v_2} \mathcal{Y}_{r_i x, s_i t}(v_4, w_4)\right] = 0 \tag{56}$$

into which the system

$$\mathcal{E}_i(V', W') - \mathcal{E}_i(V, W) = 0, \quad i = 1, 2, \tag{57}$$

and hence Eq. (54), can be decoupled on account of the relations:

$$\begin{aligned} V' - V &= v_2 - v_1 = w_3 - w_4, \quad V' + V = v_3 - v_4 = w_2 - w_1, \\ W' - W &= v_1 + v_2 = v_3 + v_4, \quad W' + W = w_1 + w_2 = w_3 + w_4. \end{aligned} \tag{58}$$

Having obtained such a system it is again easy to check whether the corresponding bilinear system (53) can be regarded as a bilinear BT. It suffices to take advantage of the linearizability of the equations which are obtained from set (56) through elimination of two pairs of mixing variables (carrying either the indices 1 and 2, 1 and 3, 1 and 4 or 3 and 4) by means of the relations:

$$v_2 = v_3 - V, \quad w_2 = w_3 + V, \quad v_4 = v_1 - V, \quad w_4 = w_1 + V. \tag{59}$$

The linearization of the equations for the two remaining pairs v_i, w_i is achieved by application of the standard map:

$$w_i = v_i + Q_i, \quad v_i = \ln \psi_i. \tag{60}$$

Thus, specializing to the case in which $i=1$ and 3 (i.e., eliminating the variables with indices 2 and 4) it follows from relations (59) and the binomial addition property [7] of $\mathscr{Y}$-polynomials, according to which:

$$e^{v_3-v_4}\mathscr{Y}_{px,qt}(v_2,w_2)=e^{v_3-v_1+V}\mathscr{Y}_{px,qt}(v_3-V,w_3+V)=e^{v_3-v_1+V}\mathscr{B}_{p,q}(v_3,w_3;V), \tag{61}$$

$$e^{v_3-v_2}\mathscr{Y}_{px,qt}(v_4,w_4)=e^{V}\mathscr{Y}_{px,qt}(v_1-V,w_1+V)=e^{V}\mathscr{B}_{p,q}(v_1,w_1;V) \tag{62}$$

with

$$\mathscr{B}_{p,q}(v_i,w_i;V)=\sum_{r=0}^{p}\sum_{s=0}^{q}(-)^{r+s}\binom{p}{r}\binom{q}{s}Y_{rx,st}(V)\mathscr{Y}_{(p-r)x,(q-s)t}(v_i,w_i) \tag{63}$$

and (see formulas (12) and (13)):

$$\mathscr{Y}_{px,qt}(v_i=\ln\psi_i,w_i=q_i+\ln\psi_i)=\psi_i^{-1}L_{p,q}(q)\psi_i,\quad i=1,3, \tag{64}$$

that each term on the left-hand side of Eq. (56) corresponds to an expression in terms of v_i, w_i, $i=1$ and 3, and V which is mapped by transformation (60) onto ψ_1^{-1} times a linear operator acting either on ψ_1 or on ψ_3, with coefficients depending on V and on the even-order derivatives of

$$Q=W-V=w_1-v_1=w_3-v_3=Q_1=Q_3. \tag{65}$$

Hence, it suffices to verify that the resulting linear equivalent of system (56)

$$\begin{aligned}\sum_i&\left[a_iL_{k_i,\ell_i}(Q)+d_ie^V\sum_{f_i=0}^{r_i}\sum_{g_i=0}^{s_i}(-)^{f_i+g_i}\binom{r_i}{f_i}\binom{s_i}{g_i}Y_{f_ix,g_it}(V)L_{r_i-f_i,s_i-g_i}(Q)\right]\psi_1\\&+\left[c_iL_{p_i,q_i}(Q)+b_ie^V\sum_{h_i=0}^{m_i}\sum_{j_i=0}^{n_i}(-)^{h_i+j_i}\binom{m_i}{h_i}\binom{n_i}{j_i}Y_{h_ix,j_it}(V)L_{m_i-h_i,n_i-j_i}(Q)\right]\psi_3\end{aligned} \tag{66}$$

is compatible under conditions on V and Q which are satisfied as a result of Eq. (52), and that it contains an arbitrary constant which plays the role of a spectral parameter. The ease with which some four-field bilinear BTs and corresponding two-component Lax systems can be obtained through this procedure is shown in the following examples.

Example 1. We consider the sine-Gordon equation

$$u_{xt}=\sin u, \tag{67}$$

which is invariant under the scale transformation:

$$x\to\lambda x,\quad t\to\lambda^{-1}t,\quad u\to u. \tag{68}$$

A dimensionless parameter c (to be determined) can be introduced by setting $u=cV$. This leads us to the equation

$$2V_{x,t}\equiv P_{x,t}(W+V)-P_{x,t}(W-V)=\frac{i}{c}\left[e^{-icV}-e^{icV}\right], \tag{69}$$

which can be decomposed over the above basis (50) if $c=2i$. Thus, it is seen that the primary version of the sine-Gordon equation

$$E(V)\equiv 4V_{xt}+e^{-2V}-e^{2V}=0 \tag{70}$$

can be decoupled into the system

$$2P_{x,t}(W+V)+e^{-2V}=0,\quad 2P_{x,t}(W-V)+e^{2V}=0. \tag{71}$$

We now consider the corresponding conditions (57):

$$2(W'-W)_{xt}-2(V'-V)_{xt}+e^{V'+V}(e^{V'-V}-e^{V-V'})=0,\quad 2(W'-W)_{xt}+2(V'-V)_{xt}+e^{-(V'+V)}(e^{V-V'}-e^{V'-V})=0, \tag{72}$$

and rewrite them in terms of the *mixing* variables (55):

$$4v_{1,xt}+e^{v_3-v_4}(e^{v_2-v_1}-e^{v_1-v_2})=0,\quad 4v_{2,xt}+e^{v_4-v_3}(e^{v_1-v_2}-e^{v_2-v_1})=0. \tag{73}$$

It is easy to see that these equations can be decoupled into a system of four equations of form (56) by the introduction of such a first-order constraint of the simplest type (aiming at the elimination of the second-order derivative in the first equation of system (73)):

$$\mathscr{Y}_x(v_1) = \lambda e^{v_3 - v_4}, \quad \lambda = \text{constant}. \tag{74}$$

This constraint reduces the first equation into

$$4\lambda(v_3 - v_4)_t + e^{v_2 - v_1} - e^{v_1 - v_2} = 0, \tag{75}$$

which can be decoupled into

$$\mathscr{Y}_t(v_3) = \frac{1}{4\lambda} e^{v_1 - v_2}, \quad \mathscr{Y}_t(v_4) = \frac{1}{4\lambda} e^{v_2 - v_1}. \tag{76}$$

The latter constraints reduce the second equation of (73) to

$$(v_{2,x} - \lambda e^{v_4 - v_3})_t = 0, \tag{77}$$

which is satisfied if

$$\mathscr{Y}_x(v_2) = \lambda e^{v_4 - v_3}. \tag{78}$$

A *linear* equivalent to the one-parameter family of first-order constraints (74), (76) and (78) can be obtained through elimination of two v_i's by means of relations (59) and the application of the map $v_j = \ln \psi_j$ on the remaining v_j's. We choose to eliminate v_1 and v_2 so as to obtain a two-component system for ψ_3 and ψ_4 which does *not* involve V_t

$$\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}_x = \begin{pmatrix} V_x & \lambda \\ \lambda & -V_x \end{pmatrix} \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}, \tag{79}$$

$$\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}_t = \begin{pmatrix} 0 & \frac{1}{4\lambda} e^{2V} \\ \frac{1}{4\lambda} e^{-2V} & 0 \end{pmatrix} \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}. \tag{80}$$

It produces a system for the combinations $\chi_1 = \frac{1}{2i}(\psi_3 - \psi_4)$ and $\chi_2 = \frac{1}{2}(\psi_3 + \psi_4)$

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda & -iV_x \\ iV_x & \lambda \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{81}$$

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}_t = \frac{1}{4\lambda} \begin{pmatrix} -\cosh 2V & -i\sinh 2V \\ -i\sinh 2V & \cosh 2V \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{82}$$

which can be recognized as the Lax system for sine-Gordon [10] upon the substitutions $\lambda = ik$ and $V = \frac{1}{2i}u$.

The bilinear equivalent of the $\mathscr{Y}$-system (74), (76) and (78):

$$D_x G' \cdot G = \lambda F' F, \quad D_t F' \cdot G = \frac{1}{4\lambda} G' F, \quad D_x F' \cdot F = \lambda G' G, \quad D_t G' \cdot F = \frac{1}{4\lambda} F' G \tag{83}$$

produces Hirota's bilinear BT for sine-Gordon [3] if one sets:

$$F = f - ig, \quad G = f + ig, \quad F' = f' - ig', \quad G' = f' + ig'. \tag{84}$$

Example 2. We now consider the modified KdV equation

$$u_t + u_{3x} + 6u^2 u_x = 0, \tag{85}$$

which is invariant under the scale transformation:

$$x \to \lambda x, \quad t \to \lambda^3 t, \quad u \to \lambda^{-1} u \tag{86}$$

indicating that $\dim(u) = -1$. A dimensionless field V can be introduced by setting $u = cV_x$, with c = dimensionless parameter (to be determined). This leads, after one integration with respect to x, to the potential equation

$$V_t + V_{3x} + 2c^2 V_x^3 = 0, \tag{87}$$

which can be expressed in terms of $\mathscr{Y}$-polynomials upon an appropriate choice of c ($c^2 = -1$)

$$\mathscr{Y}_t(V) + \mathscr{Y}_{3x}(V, W) - 3\mathscr{Y}_x(V)\mathscr{Y}_{2x}(V, W) = 0. \tag{88}$$

It is clear that Eq. (88) can be decoupled into the system:

$$\mathscr{Y}_{2x}(V,W)=0,\quad \mathscr{Y}_t(V)+\mathscr{Y}_{3x}(V,W)=0. \tag{89}$$

Conditions (57) associated with this system are:

$$(W'-W)_{2x}+(V'-V)_x(V'+V)_x=0,\quad (V'-V)_t+(V'-V)_{3x}+3(V'_xW'_{2x}-V_xW_{2x})+(V'^3_x-V^3_x)=0. \tag{90}$$

We rewrite them in terms of the mixing variables v_i, w_i:

$$\begin{aligned}&(v_3+v_4)_{2x}+(v_2-v_1)_x(v_3-v_4)_x=0,\\&(v_2-v_1)_t+(v_2-v_1)_{3x}+\frac{1}{4}(v_2-v_1)_x[3(v_{3,x}-v_{4,x})^2+(v_{2,x}-v_{1,x})^2+6(w_1+w_2)_{2x}]+\frac{3}{2}(v_3-v_4)_x(v_3+v_4)_{2x}=0\end{aligned} \tag{91}$$

and notice that the first condition is satisfied if we impose the first-order constraints:

$$\mathscr{Y}_x(v_3)=\lambda e^{v_1-v_2},\quad \mathscr{Y}_x(v_4)=\mu e^{v_2-v_1},\qquad \lambda,\mu=\text{constant}. \tag{92}$$

The second condition can then be rewritten as follows:

$$\mathscr{Y}_t(v_2)+\mathscr{Y}_{3x}(v_2,w_2)-\mathscr{Y}_t(v_1)-\mathscr{Y}_{3x}(v_1,w_1)\mathscr{Y}_t(v_2)+\mathscr{Y}_{3x}(v_2,w_2)-\mathscr{Y}_t(v_1)-\mathscr{Y}_{3x}(v_1,w_1)+3\lambda\mu[\mathscr{Y}_x(v_2)-\mathscr{Y}_x(v_1)]=0. \tag{93}$$

It is clear that this last condition can be decoupled into the following pair of bilinearizable equations:

$$\mathscr{Y}_t(v_1)+\mathscr{Y}_{3x}(v_1,w_1)+3\lambda\mu\mathscr{Y}_x(v_1)=0,\quad \mathscr{Y}_t(v_2)+\mathscr{Y}_{3x}(v_2,w_2)+3\lambda\mu\mathscr{Y}_x(v_2)=0. \tag{94}$$

Eliminating v_3 and v_4 from Eqs. (92) by means of relations (59) one finds that the system consisting of Eqs. (94) and

$$\mathscr{Y}_x(v_2)=-V_x+\lambda e^{v_1-v_2},\quad \mathscr{Y}_x(v_1)=V_x+\mu e^{v_2-v_1} \tag{95}$$

is mapped by the transformation $v_j=\ln\psi_j$, with $j=1$ and 2, onto the linear two component system:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}V_x & \mu\\ \lambda & -V_x\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \tag{96}$$

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t=\begin{pmatrix}A & B\\ C & -A\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{97}$$

with

$$\begin{aligned}&A=-\mathscr{Y}_{3x}(V,W)-4\lambda\mu\mathscr{Y}_x(V),\\&B=-\mu[\mathscr{Y}_{2x}(V,W)+2P_{2x}(W-V)+4\lambda\mu],\\&C=-\lambda[\mathscr{Y}_{2x}(V,W)+2P_{2x}(W+V)+4\lambda\mu].\end{aligned} \tag{98}$$

The integrability of this system is subject to the following conditions on V and W (assuming that $\lambda\mu\neq 0$):

$$\partial_x\mathscr{Y}_{2x}(V,W)=0,\quad \partial_x[\mathscr{Y}_t(V)+\mathscr{Y}_{3x}(V,W)]=0, \tag{99}$$

both of which are satisfied as a result of Eqs. (89).

It follows that the *bilinear equivalent* of system (96) and (97)

$$\begin{aligned}&D_xF'\cdot G=\lambda GF',\\&D_xG'\cdot F=\mu F'G,\\&\left[D_t+D_x^3+3\lambda\mu D_x\right]G'\cdot G=0,\\&\left[D_t+D_x^3+3\lambda\mu D_x\right]F'\cdot F=0\end{aligned} \tag{100}$$

constitutes a *bilinear BT* for the modified KdV equation (85).

Symmetry of the bilinear BT under the interchange $F\rightleftharpoons G$, $F'\rightleftharpoons G'$ (which corresponds to the symmetry of PmKdV under the map $V\to -V$) can be obtained by choosing $\lambda=\mu$. This *symmetric* bilinear BT coincides with the one given by Hirota [3] after the substitution of (84). Eliminating W_{2x} from the two-component system (96) and (97), by means of the constraint $\mathscr{Y}_{2x}(V,W)=0$ at $\lambda=\mu$, one finds a linear system for the combinations $\chi_1=\frac{1}{2}(\psi_1+\psi_2)$ and $\chi_2=\frac{1}{2}(\psi_1-\psi_2)$

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_x=\begin{pmatrix}\lambda & V_x\\ V_x & -\lambda\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \tag{101}$$

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_t=\begin{pmatrix}-4\lambda^3+2\lambda V_x^2 & -4\lambda^2V_x-2\lambda V_{2x}+2V_x^3-V_{3x}\\ -4\lambda^2V_x+2\lambda V_{2x}+2V_x^3-V_{3x} & +4\lambda^3-2\lambda V_x^2\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \tag{102}$$

which coincides with the Lax system for the modified KdV given in [11] upon the substitution $\lambda=-\mathrm{i}\zeta$.

Example 3. The AKNS-system

$$\mathrm{i}\psi_t+\psi_{2x}+2\psi^2\widehat{\psi}=0,\quad -\mathrm{i}\widehat{\psi}_t+\widehat{\psi}_{2x}+2\psi\widehat{\psi}^2=0 \tag{103}$$

is mapped by the transformation $\psi=\mathrm{e}^V,\ \widehat{\psi}=\mathrm{e}^{\widehat{V}}$ (aiming at the elimination of nonexponential factors without derivatives which do not appear in the basis (50)) onto:

$$\mathrm{i}V_t+V_{2x}+V_x^2+2\mathrm{e}^{V+\widehat{V}}=0,\quad -\mathrm{i}\widehat{V}_t+\widehat{V}_{2x}+\widehat{V}_x^2+2\mathrm{e}^{V+\widehat{V}}=0 \tag{104}$$

or the equivalent system

$$\mathrm{i}V_t+V_{2x}+V_x^2+2\mathrm{e}^{V+\widehat{V}}=0,\quad V_{2t}+V_{4x}-2\mathrm{i}(V_{2x}V_t+2V_xV_{xt})-6V_x^2V_{2x}=0. \tag{105}$$

The first equation can be rewritten as a linear combination of $\mathscr{Y}$-polynomials

$$\mathrm{i}\mathscr{Y}_t(V)+\mathscr{Y}_{2x}(V,W)=0 \tag{106}$$

upon the introduction of an auxiliary W-field, subject to the constraint

$$W_{2x}=V_{2x}+2\mathrm{e}^{V+\widehat{V}.} \tag{107}$$

The second equation can also be rewritten in terms of such combinations and takes the form

$$\partial_t[\mathscr{Y}_t(V)-\mathrm{i}\mathscr{Y}_{2x}(V,W)]+\partial_x\{\mathscr{Y}_{3x}(V,W)+\mathrm{i}\mathscr{Y}_{x,t}(V,W)-3\mathscr{Y}_x(V)[\mathrm{i}\mathscr{Y}_t(V)+\mathscr{Y}_{2x}(V,W)]\}=0. \tag{108}$$

It follows that system (104) can be decoupled into the following system for V and W:

$$\begin{array}{l}\mathrm{i}\mathscr{Y}_t(V)+\mathscr{Y}_{2x}(V,W)=0,\\ \mathrm{i}\mathscr{Y}_{x,t}(V,W)+\mathscr{Y}_{3x}(V,W)=0,\end{array}\iff\begin{array}{l}\mathrm{i}V_t+W_{2x}+V_x^2=0,\\ \mathrm{i}W_{x,t}+V_{3x}+2V_xW_{2x}=0.\end{array} \tag{109}$$

Conditions (57) associated with this system

$$\begin{array}{l}\mathrm{i}(V'-V)_t+(W'-W)_{2x}+(V'-V)_x(V'+V)_x=0,\\ \mathrm{i}(W'-W)_{xt}+(V'-V)_{3x}+(V'-V)_x(W'+W)_{2x}+(V'+V)_x(W'-W)_{2x}=0\end{array} \tag{110}$$

can be rewritten in terms of the mixing fields v_i, w_i

$$\begin{array}{l}\mathrm{i}(v_2-v_1)_t+(w_1-w_2)_{2x}+2v_{3,2x}+(v_{1,x}^2-v_{2,x}^2)+2v_{3,x}(v_2-v_1)_x=0,\\ \mathrm{i}(v_1+v_2)_{xt}+(v_2-v_1)_{3x}+(v_2-v_1)_x(w_1+w_2)_{2x}+(2v_3-v_1-v_2)_x(v_1+v_2)_{2x}=0,\end{array} \tag{111}$$

where we have used the relations: $v_1+v_2=w_1-w_2+2v_3,\ v_3-v_4=2v_3-v_1-v_2$. The first equation can be decoupled into the following bilinearizable constraints ($\mu=$constant):

$$\begin{array}{l}\mathrm{i}\mathscr{Y}_t(v_1)-\mathscr{Y}_{2x}(v_1,w_1)=0,\\ \mathrm{i}\mathscr{Y}_t(v_2)-\mathscr{Y}_{2x}(v_2,w_2)=0,\\ \mathscr{Y}_x(v_3)=\mu\mathrm{e}^{v_1-v_2}.\end{array} \tag{112}$$

Differentiating the two first equations of the latter system with respect to x and using the relation $w_1+w_2+v_2-v_1=2w_3$ it is seen that the second equation of system (111) can be recast in the form

$$w_{3,3x}+(v_2-v_1)_xw_{3,2x}+v_{3,x}(v_3+v_4)_{2x}=0, \tag{113}$$

which, on account of the last equation of (112), amounts to the condition

$$v_{3,x}\left(\frac{w_{3,2x}}{v_{3,x}}\right)_x = -v_{3,x}(v_3+v_4)_{2x}. \tag{114}$$

This condition is satisfied if one requires that ($v=$constant)

$$\frac{w_{3,2x}}{v_{3,x}} + (v_3+v_4) = v \quad \Longleftrightarrow \quad \mathscr{Y}_{2x}(v_3,w_3) - v\mathscr{Y}_x(v_3) + v_{3,x}v_{4,x} = 0. \tag{115}$$

In view of the last equation of system (112), it is found that the latter takes the form

$$\mathscr{Y}_{2x}(v_3,w_3) - v\mathscr{Y}_x(v_3) + \mu e^{v_1-v_2}\mathscr{Y}_x(v_4) = 0. \tag{116}$$

Hence, it turns out that conditions (111) can be decoupled into the four bilinearizable constraints (112) and (116) which, after elimination of the mixing fields with indices 2 and 4 by means of relations (59), and application of the transformation $v_i = \ln\psi_i$ with $i=1$ and 3, are mapped onto the following linear two component system in terms of V and $\widehat{V}$:

$$\begin{pmatrix}\psi_1\\ \psi_3\end{pmatrix}_x = \begin{pmatrix}\frac{v}{2} & -\frac{1}{\mu}e^{\widehat{V}}\\ \mu e^{V} & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_3\end{pmatrix}, \tag{117}$$

$$i\begin{pmatrix}\psi_1\\ \psi_3\end{pmatrix}_t = \begin{pmatrix}\frac{v^2}{4}+e^{V+\widehat{V}} & -\frac{1}{\mu}\left(\frac{v}{2}+\widehat{V}_x\right)e^{\widehat{V}}\\ \mu e^{V}\left(\frac{v}{2}-V_x\right) & iV_t+V_{2x}+V_x^2+e^{V+\widehat{V}}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_3\end{pmatrix}, \tag{118}$$

where we have used relations (58) and the first equation of (109).

The compatibility of this system is subject to conditions on V and $\widehat{V}$ (independent of μ and v) which are found to reduce to the primary AKNS-system (104).

Setting $v=-4i\zeta,\ \psi_1=(1/\mu)\phi\chi_2,\ \psi_2=\phi\chi_1$ with $\phi=e^{-i\zeta x+2i\zeta^2 t}$ and taking account of the the first equation of (109) it is found that the coupled AKNS-system can be regarded as the integrability condition for the system:

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_x = \begin{pmatrix}-i\zeta & \psi\\ \widehat{\psi'} & i\zeta\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \tag{119}$$

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_t = \begin{pmatrix}-2i\zeta^2 - i\psi\widehat{\psi'} & i\psi_x+2\zeta\psi\\ 2\zeta\widehat{\psi'} - i\widehat{\psi'}_x & 2i\zeta^2+i\psi\widehat{\psi'}\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \tag{120}$$

which is the linear system reported in [11].

The bilinear equivalent of system (112) and (116):

$$\begin{aligned}&[iD_t - D_x^2]G'\cdot G = 0,\\ &[iD_t - D_x^2]F'\cdot F = 0,\\ &D_xF'\cdot G = \mu FG',\\ &[D_x^2 - vD_x]F'\cdot G + \mu D_xG'\cdot F = 0\end{aligned} \tag{121}$$

constitutes a bilinear BT for the AKNS-system.

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