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An upper bound on the cycle time of a stochastic marked graph using incomplete information on the transition firing time distributions

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ABSTRACT

Stochastic marked graphs, a special class of stochastic timed Petri nets, are used for modelling and analyzing decision-free dynamic systems with uncertainties in timing. The model allows evaluating the performance of such systems under a cyclic process. Given the probabilistic characteristics of the transition times, the cycle time of the system can be determined from the initial marking. In this contribution, we compute an upper bound on the cycle time of a stochastic marked graph in case the probabilistic characteristics of the transition times are not fully specified.

1. Introduction

Petri nets are an established model to represent and analyze concurrent systems. A Petri net is a collection of directed arcs connecting places and transitions. These arcs have a default capacity of one unless stated otherwise. Places can contain tokens, and the assignment of tokens to places is called the state or marking of the net. Arcs can only connect places to transitions and vice versa. A transition is said to be enabled if the number of tokens in its input places is at least equal to the arc weight going from the input places to the transition. Once enabled, a transition can fire. When fired, the tokens in the input places are moved to output places, according to the arc weights and place capacities.

A marked graph is a Petri net in which each place has at most one input transition and one output transition. Marked graphs constitute a good formalism to model manufacturing systems containing parallel tasks and synchronization or to order activities like in PERT. They are more general than PERT graphs in the sense that places can contain several tokens. Marked graphs have been studied extensively either in a deterministic or in a stochastic context. One of the main problems of timed and stochastic Petri net models for large systems is the explosion of computational complexity algorithms to analyze performance measures (such as the cycle time) of marked graphs. Campos et al. [4] determine upper and lower bounds on the steady-state performance of marked graphs to evaluate performance in an efficient way. In this paper we develop a tight upper bound for the cycle time of a stochastic marked graph.

The remainder of this paper is structured as follows. Notations and definitions are introduced in Section 2. A lower bound on the cycle time of a stochastic marked graph is presented in Section 3. In Section 4 a tighter upper bound than the one in [4] is developed. Section 5 develops the bound in the case that incomplete information exists regarding the distribution of the transition firing times.

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2. Notations and definitions

Petri nets are developed to represent and analyze general systems from the viewpoint of causal interconnections among the elementary system components. There are several types or classes of Petri nets. Some basic classes are: condition/event, place/transition and predicate/transition nets.

The common conceptual basis for all Petri net classes is a 'net'. A net is a bipartite digraph together with a weak interpretation. The interpretation is called 'weak' because any use of the net requires further interpretation.

A net is a triple $N = (S, T, F)$ which represents a system with S-elements (S is a finite set of which the elements represent state elements), with T-elements (T is a finite set of which the elements represent transition elements). The elements $S \cup T$ are called the elements of the net. The S-elements are represented by circles; the T-elements by vertical bars or boxes. The relations between S- and T-elements are combined yielding the flow relation F, $F \subseteq (S \times T) \cup (T \times S)$. F is a finite set of ordered pairs of two types (S-element, T-element) and (T-element, S-element) representing causal connections. The elements of F are represented by arcs. S and T are disjoint sets [16, chapter 1].

In a *condition/event net* (CE-net) the S-elements represent elementary system conditions, while its T-elements represent elementary system events. A dot (called a token) is put into an S-element if the condition corresponding to this S-element is met, which implies that every S-element can have either zero or one token. In other nets the S-elements may carry more than one token. In such nets S-elements are called *places*, the T-elements are called *transitions* and weights can be attached to the arcs. The weights are natural numbers. Each place has a capacity (maybe infinite), representing the maximum number of tokens a place can hold. This type of net is called a Place/Transition or P/T net.

A Place/Transition (P/T) net is defined as a 3-tuple $N = (P, T, F)$ consisting of a set of places P, a set of transitions T and directed arcs connecting places to transitions and transitions to places, $F \subseteq (P \times T) \cup (T \times P)$. Let $|P| = n$ (resp. $|T| = m$) be the cardinality of the set of places (resp. transitions). Further, assume $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$. The input arcs can be represented by the pre-incidence function $Pre : P \times T \to \mathbb{N}$. $Post$ is the post-incidence function representing the output arcs $Post : P \times T \to \mathbb{N}$. A P/T net is an ordinary P/T net if its pre- and post-incidence functions take values in $\{0, 1\}$.

The pre- and post-incidence functions can be represented as $n \times m$ matrices Pre and $Post$ with elements $Pre(p_i, t_j)$ and $Post(p_i, t_j)$ resp. The incidence matrix C of the net is defined by $C(p_i, t_j) = Post(p_i, t_j) - Pre(p_i, t_j)$.

The set of places connected to a transition via a directed arc from the place to the transition is called the pre-set of the transition (similar for places). The set of places connected to a transition via a directed arc from the transition to the place is called the post-set of the transition (similar for places). Formally, the pre-set and post-set of a transition $t \in T$ are defined resp. as $\bullet t = \{p | Pre(p, t) > 0\}$ and $t\bullet = \{p | Post(p, t) > 0\}$. The pre-set and post-set of a place $p \in P$ are defined resp. as $\bullet p = \{t | Post(p, t) > 0\}$ and $p\bullet = \{t | Pre(p, t) > 0\}$.

A marking $M : P \to \mathbb{N}$ assigns tokens to places. It can be represented as a vector, i.e. M_i represents the number of tokens in place i in marking M. A marked net (N, M_0) is a net with an initial marking M_0.

A transition is enabled (i.e. can possibly occur) if the number of tokens in its input places is at least equal to the weight of the arcs going from the input places to the transition. Once enabled, a transition can fire. When fired, the tokens in the input places are moved to the output places, according to the arc weights. Formally, a transition $t \in T$ is enabled at marking M iff $\forall p \in P : M(p) \geq Pre(p, t)$. A transition t enabled at M can fire, yielding a new marking M' based on the incidence matrix C, $M'(p) = M(p) + C(p, t), \forall p \in P$.

The reachability set $R(N, M_0)$ is the set of all markings reachable from the initial marking M_0. Liveness considers which transitions in a net can fire and how often transitions may fire. A transition $t \in T$ is live in (N, M_0) iff $\forall M \in R(N, M_0) : \exists M' \in R(N, M)$ such that M' enables t. A marked net (N, M_0) is live iff all its transitions are live.

A P-semiflow (or P-invariant) is a vector $Y \in \mathbb{N}^n$ such that $Y \neq 0$ and $Y^T C = 0$. The support of a P-semiflow Y is the set of places which appear in Y, i.e. whose corresponding components in Y are strictly positive: $\|Y\| = \{p \in P | Y(p) > 0\}$. A support is a minimal support iff it does not contain another support or invariant but itself and the empty set.

This paper focuses on marked graphs, which are a special case of marked nets. A P/T net is called a marked net iff, for all $s \in P : K_s = \infty, M_s \in N$, and for all $f \in F, W_f = 1$. This means that all arcs have unit weight and that we do not have to take capacity into consideration. In a marked graph, places are unbranched, i.e. $\forall s \in P : |\bullet s| = |s \bullet| = 1$. A sequence $w = (s_0, \ldots, s_n)$ of places is called a path of length n. w is called a cycle or circuit iff w is a path such that $\bullet s_0 = s_n$ [16, chapter 7]. A marked graph is said to be *strongly connected* if there is a directed path joining any node A to any node B of the net. An *elementary circuit* in a strongly connected graph is defined as a directed path that goes from one node (place or transition) back to the same node while any other node is not repeated.

As an illustration, consider Fig. 1, taken from [10], depicting a marked graph with $|P| = 14$ and $|T| = 10$. The marked graph is strongly connected because there is a direct path joining any node to any other node in the graph. The minimal support P-semiflows of N are its directed elementary circuits. The marked net (N, M_0) is live iff all its directed circuits are marked [12]. As a result, the directed elementary circuits or P-semiflows in Fig. 1 are $\|Y_1\| = \{p_1, p_2, p_4\}$, $\|Y_2\| = \{p_3, p_5, p_6, p_{14}\}$, $\|Y_3\| = \{p_7, p_8, p_9, p_{10}\}$, $\|Y_4\| = \{p_3, p_4, p_8, p_9, p_{11}, p_{13}, p_{14}\}$, $\|Y_5\| = \{p_7, p_8, p_9, p_{11}, p_{12}\}$. The marked graph is live if, for example, one token is placed in the places p_1, p_6, p_{10}, p_{12}, and p_{13}. In this case, each of the elementary circuits contains exactly one token.

Using and extending Petri nets by adding time features is of high interest in many fields, especially in the field of performance evaluation. Different choices for the integration of time into the Petri net formalism have been proposed in the literature. Two basic Petri net models for handling time have been proposed: Time Petri nets and Timed Petri nets.

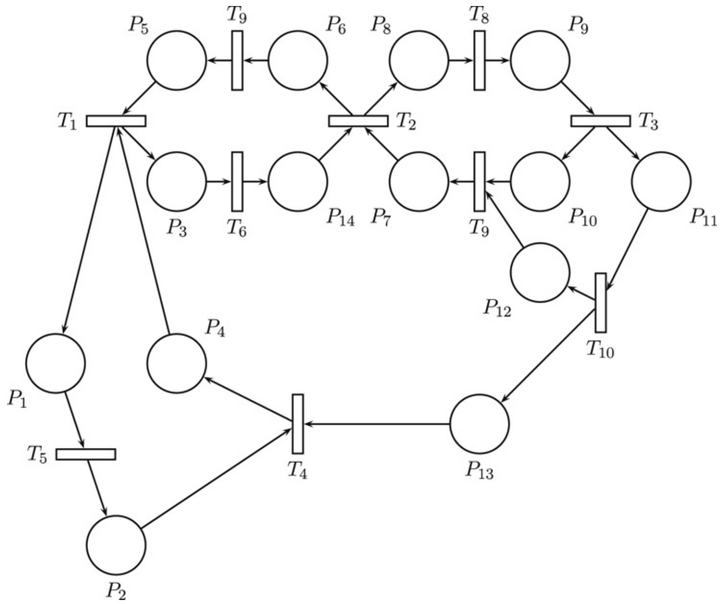

Fig. 1. An example of a stochastic marked graph.

Timed Petri nets are derived from Petri nets by associating a finite firing duration with each transition of the net [15] or with each place of the net [18]. Time Petri nets [11] are more general. Two real numbers, a and b, with $a \leq b$, are associated with each transition where a is the minimal time that must elapse, starting from the time at which a transition is enabled, until this transition can fire, and b denotes the maximum time during which the transition can be enabled without being fired. Timed marked graphs are timed Petri nets whose logical structure is that of a marked graph. Regarding the introduction of time in this paper, we associate firing times with transitions.

In this paper we consider the performance evaluation of stochastic timed transition marked graphs. For a large net the computational complexity of the analysis algorithms is a problem. If the firing time distributions are assumed to follow exponential distributions, the equivalence between stochastic Petri nets and homogeneous Markov processes has been established. In this situation performance measures can be derived using Markovian analysis techniques. In the case of general distributions, the calculation of performance measures becomes intractable. In such a case upper and lower bounds on performance measures offer some information, see e.g. [4]. If information regarding the firing time distributions is incomplete, e.g. limited to some moments, these bounds are the only means to provide useful information. We derive upper and lower bounds for the mean cycle time of a stochastic marked graph using the knowledge of the first and second moments of the distribution only.

Even in the case the transition firing times follow exponential distributions, as with any other stochastic Petri net, the numerical analysis of the continuous time Markov chain (CTMC) underlying the SMG suffers from the state space explosion problem, which excludes applicability for large models. Approximate analysis techniques have been used either to derive results where exact techniques are too costly or not applicable at all, or to derive a good initial distribution for exact iterative techniques [3].

As another way to cope with state explosion, fluid approximation has been introduced to reduce the number of states. In a fluid/event graph, places hold fluids instead of discrete tokens. Transitions fire continuously, drawing fluids out of its input places and injecting fluids into its output places. Sometimes to these models discrete-event components characterized by failures and repair of transitions are added [5]. Only in very specific cases analytical solutions for the performance evaluation can be obtained. Mostly a simulation-based approach is being used [19].

Denote the random variable generating the time required for the kth firing of transition t by $Y_t(k) \in \mathbb{R}^+$ and the instant of the kth firing initiation of transition t by $S_t(k)$. Assume further that no transition can be fired by more than one token at any given time. We further assume that, when a transition fires, the related tokens remain in the input places until the firing process ends. Immediately after the firing, these tokens disappear and the appropriate number of tokens is added to each output place of the transition.

We assume that the sequences of transition firing times $\{Y_t(k)\}_{k=1}^{\alpha}$ for $t \in T$ are mutually independent sequences of independent identically distributed (i.i.d.) random variables. Since $\{Y_t(k)\}$ are sequences of i.i.d. random variables, the index k is often abandoned. Furthermore, we denote $m_t = \mathrm{E}[Y_t]$, and $\sigma_t^2 = \mathrm{E}\left[(Y_t - m_t)^2\right]$.

Under the foregoing assumptions, it is proven [1] that there exists a positive constant $\pi(M_0)$ such that

$$\lim_{k\to\infty} \frac{S_t(k)}{k} = \lim_{k\to\infty} \frac{\mathrm{E}[S_t(k)]}{k} = \pi(M_0) \quad \forall t \in T \tag{1}$$

where $\pi(M_0)$ is called the cycle time of the marked graph.

Table 1
Transitions and deterministic firing times for Fig. 1

Transition	Firing time	Transition	Firing time
t_1	9.10	t_6	4.60
t_2	8.20	t_7	3.70
t_3	7.30	t_8	2.80
t_4	6.40	t_9	1.90
t_5	5.50	t_{10}	1.45

Table 2
Circuits and circuit times for Fig. 1

Circuit	Transitions in circuit	Circuit time
Y_1	t_1, t_4, t_5	21.00
Y_2	t_1, t_2, t_6, t_7	25.60
Y_3	t_2, t_3, t_8, t_9	20.20
Y_4	$t_1, t_2, t_3, t_4, t_6, t_8, t_{10}$	39.85
Y_5	$t_2, t_3, t_8, t_9, t_{10}$	21.65

The cycle time of a deterministic marked graph $(N, M_0, \{Y_t\})$ is denoted by $\pi^D(M_0)$ and is equal to [14, p. 443]:

$$\pi^D(M_0) = \lim_{k\to\infty} \frac{S_t(k)}{k} = \max_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} \tag{2}$$

where Γ is the set of all circuits in the graph.

The cycle time for a deterministic marked graph is the maximum cycle time over all elementary circuits $\gamma \in \Gamma$. The cycle time of an individual elementary circuit γ is defined as the sum over the firing times of each transition in the circuit divided by the circuit's initial marking $M_0(\gamma)$.

Since $\pi(M_0) \geq \pi^D(M_0)$, $\pi^D(M_0)$ is a lower bound on $\pi(M_0)$.

3. A lower bound on the cycle time of a stochastic marked graph

Prior to determining the lower bound for a stochastic marked graph, the bound is determined for its deterministic counterpart. In a deterministic marked graph a firing function $\mathcal{T} : T \to \mathbb{Q}^+$ is defined where $\mathbb{Q}^+$ is the set of non-negative rational numbers. It has been proven that the minimum cycle time of a deterministic timed marked graph is given by [14]

$$\pi^D(M_0) = \max_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} \tag{3}$$

where m_t is the deterministic firing time of transition t, and $M_0(\gamma)$ is the number of tokens in the places in circuit γ in the initial marking.

Determining the minimal cycle time has been formulated as a linear programming problem [9] which uses the following relation: the termination time of the nth firing of transition $t_i \leq$ the initiation time of the $(n + M_0(p_k))$th firing of transition t_j if t_i is the input transition and t_j is the output transition of place p_k.

This means that, for any cycle time CT,

$$S_{t_i}(n) + m_{t_i} \leq S_{t_j}(n + M_0(p_k)) \tag{4}$$

or

$$m_{t_i} \leq S_{t_j}(1) - S_{t_i}(1) + M_0(p_k).CT \tag{5}$$

By this, the linear program (LP) can be formulated as:

$$\begin{aligned} &\min\ CT \\ &\text{s.t.}\quad m_{t_i} \leq S_{t_j}(1) - S_{t_i}(1) + M_0(p_k).CT \quad \forall p_k \in P \text{ and } CT \geq 0 \end{aligned} \tag{6}$$

where $S_{t_1}(1), S_{t_2}(1), \ldots, S_{t_{|T|}}(1)$ and CT are the decision variables of the optimization problem. The solution of this linear program using the deterministic transition times from Fig. 1 as shown in Table 1 results in the optimal value of the objective function $CT = 39.85$.

In Section 1 five elementary circuits have been detected in Fig. 1. According to [14] the circuit times, i.e. the sum of the firing times of the transitions in a circuit, have to be inspected for their maximum value. Detailed information on the circuit times is given in Table 2.

Following [9] the minimal cycle time can be found as the solution to the linear program (LP1) stated as:

$$\begin{aligned}
\min\ & CT \\
\text{s.t.}\ & S_{t_5}(1) - S_{t_1}(1) + CT \geq 9.10 \\
& S_{t_4}(1) - S_{t_5}(1) \geq 5.50 \\
& S_{t_6}(1) - S_{t_1}(1) \geq 9.10 \\
& S_{t_1}(1) - S_{t_4}(1) \geq 6.40 \\
& S_{t_2}(1) - S_{t_7}(1) \geq 3.70 \\
& S_{t_7}(1) - S_{t_2}(1) + CT \geq 8.20 \\
& S_{t_2}(1) - S_{t_9}(1) \geq 1.90 \\
& S_{t_8}(1) - S_{t_2}(1) \geq 8.20 \\
& S_{t_3}(1) - S_{t_8}(1) \geq 2.80 \\
& S_{t_9}(1) - S_{t_3}(1) + CT \geq 2.80 \\
& S_{t_{10}}(1) - S_{t_3}(1) \geq 7.30 \\
& S_{t_9}(1) - S_{t_{10}}(1) + CT \geq 7.30 \\
& S_{t_4}(1) - S_{t_{10}}(1) + CT \geq 1.45 \\
& S_{t_2}(1) - S_{t_6}(1) \geq 4.60
\end{aligned} \tag{7}$$

which gives an optimal objective function value $CT = 39.85$.

When switching from deterministic marked graphs towards stochastic marked graphs a similar linear optimization program has been formulated by [4, p. 390]. A lower bound for the mean cycle time for live strongly connected marked graphs can be obtained by solving the following linear program:

$$\begin{aligned}
\pi(M_0) = \max\ & Y^T.Pre.\theta \\
\text{s.t.}\ & Y^T.C = 0 \\
& Y^T.M_0 = 1 \\
& Y \geq 0
\end{aligned} \tag{8}$$

with

$Y =$ a P-semiflow or P-invariant
$\theta =$ a vector with components $\theta_t = E(Y_t),\ t \in T$
$C =$ the incidence matrix
$M_0 =$ the initial marking of the graph.

As deterministic timed graphs are a special case of stochastic marked graphs with the mean transition firing time equal to the deterministic firing times, both LPs give the same solution in the case of deterministic marked graphs [4, pp. 391–392]. Furthermore, Campos et al. [4] prove that for strongly connected marked graphs with arbitrary values of mean and variance for transition firing times, the lower bound for the mean cycle time obtained in the above LP cannot be improved (their theorem 3.3).

If both mean and variance of the firing time of each transition are known, the lower bound as obtained by the LP cannot be reached (unless all variances are equal to zero). This lower bound is, however, not our main interest. The next section focuses on the upper bound for the mean cycle time. For its calculation we require the value of the lower bound in the case of deterministic firing times.

4. An upper bound on the cycle time of a stochastic marked graph with complete information on the stochastic transition times

In the previous section a lower bound of the average cycle time has been derived and it is shown that the P-invariant criterion reaches its minimum value when the firing times become deterministic. In this section upper bounds of the average cycle time are derived making use of superposition properties.

Sauer and Xie [17] prove that the following bound holds:

$$\pi(M_0) \leq \pi^D(M_0) + \inf_{\mathbf{z}\in E}\left\{\sum_{t\in T} \mathrm{E}\left[(X_t - z_t)^+\right]\right\} \tag{9}$$

where the infimum needs to be found in the set E defined as:

$$E = \left\{\mathbf{z} | z_t \geq m_t, \forall t \in T \text{ and } \sum_{t\in\gamma} z_t \leq \pi^D(M_0).M_0(\gamma), \forall \gamma \in \Gamma\right\}, \mathbf{z} = [z_1, z_2, \ldots, z_m]. \tag{10}$$

Using the first two moments of the random variables representing the transition firing times, Theorem 1 shows that the upper bound obtained by Sauer and Xie [17]

$$\pi(M_0) \leq \pi^D(M_0) + \sum_{t\in T} \sigma_t \tag{11}$$

can be improved.

Theorem 1.

$$\pi(M_0) \leq \pi^D(M_0) + \sum_{t\in T} \frac{\sigma_t}{2}.$$

Proof. It always holds that

$$\inf_{\mathbf{z}\in E} \left\{ \sum \mathrm{E}\left[(X_t - z_t)^+\right] \right\} \leq \sum_{t\in T} \mathrm{E}\left[(X_t - m_t)^+\right]. \tag{12}$$

Moreover, the following equality holds for any t:

$$\mathrm{E}\left[(X_t - t)^+\right] = \frac{\mathrm{E}\left[|X_t - t|\right] + \mathrm{E}\left[X_t\right] - t}{2}. \tag{13}$$

So by putting in equality (13) $t = m_t$ and because $\mathrm{E}[X_t] = m_t$

$$\mathrm{E}\left[(X_t - m_t)^+\right] = \frac{\mathrm{E}\left[|X_t - m_t|\right]}{2}. \tag{14}$$

By this, inequality (12) can be rewritten as

$$\begin{aligned} \inf_{\mathbf{z}\in E} \left\{ \sum \mathrm{E}\left[(X_t - z_t)^+\right] \right\} &\leq \frac{1}{2} \sum_{t\in T} \mathrm{E}\left[|X_t - m_t|\right] \\ &\leq \frac{1}{2} \sum_{t\in T} \sqrt{\mathrm{E}\left[X_t - m_t\right]^2} \\ &\leq \sum_{t\in T} \frac{\sigma_t}{2}. \quad \square \end{aligned} \tag{15}$$

In order to assess the evolution of the right-hand sides from $\mathrm{E}\left[(X_t - m_t)^+\right]$ in inequality (12) to $\frac{\sigma_t}{2}$ in inequality (15), we illustrate the procedure in the case where X_t follows a triangular distribution.

Let $f_t(x)$ be the density function of a triangular distribution of the firing times of a transition t defined on the finite interval $[a_t, b_t]$ with mode located at c_t:

$$f_t(x) = \begin{cases} f_{1t}(x) = \dfrac{2(x - a_t)}{(c_t - a_t)(b_t - a_t)} & \text{if } a_t \leq x \leq c_t \\ f_{2t}(x) = \dfrac{2(b_t - x)}{(b_t - c_t)(b_t - a_t)} & \text{if } c_t \leq x \leq b_t. \end{cases} \tag{16}$$

The mean, respectively the variance of this distribution are:

$$m_t = \frac{a_t + c_t + b_t}{3} \tag{17}$$

$$\sigma_t^2 = \frac{a_t^2 + b_t^2 + c_t^2 - a_t b_t - a_t c_t - b_t c_t}{18}. \tag{18}$$

The illustration is applied to the stochastic marked graph of Fig. 1. Table 3 shows the parameters of the triangular transition time distribution for each transition. The mean values of the distributions correspond to the deterministic values which were used earlier (see Table 1).

If X_t is distributed according to a triangular distribution, we are able to calculate the right-hand side in inequality (12). The value of $\pi^D(M)$ is independent of the knowledge of the distribution. Therefore, to assess the difference between the bound with incomplete information and the one with complete information, only the right part of inequality (12) needs to be calculated.

The right-hand side of inequality (12) is computed as

$$\mathrm{E}\left[(X_t - m_t)^+\right] = \int_{m_t}^{b_t} (x - m_t).f(x)\mathrm{d}x \tag{19}$$

Table 3
Data for the triangular distributions of the example in Fig. 1

Transition	a_t	c_t	b_t	Mean	Variance	St. Dev.
t_1	1	12.3	14	9.1	8.322	2.885
t_2	1	10.6	13	8.2	6.720	2.592
t_3	1	8.9	12	7.3	5.362	2.316
t_4	1	7.2	11	6.4	4.247	2.061
t_5	1	5.5	10	5.5	3.375	1.837
t_6	1	3.8	9	4.6	2.747	1.657
t_7	1	2.6	7.5	3.7	1.912	1.383
t_8	1	1.4	6	2.8	1.287	1.134
t_9	1	0.7	4	1.9	0.555	0.745
t_{10}	1	0.35	3	1.45	0.318	0.564

Table 4
Bounds for the example in Fig. 1

a_t	c_t	b_t	Mean	Variance	Exact	Bound	RelDiff
1	12.3	14	9.100	8.322	1.206	1.442	0.196
1	10.6	13	8.200	6.720	1.080	1.296	0.200
1	8.9	12	7.300	5.362	0.959	1.158	0.207
1	7.2	11	6.400	4.247	0.847	1.030	0.217
1	5.5	10	5.500	3.375	0.750	0.919	0.225
1	3.8	9	4.600	2.747	0.683	0.829	0.214
1	2.6	7.5	3.700	1.912	0.574	0.691	0.204
1	1.4	6	2.800	1.287	0.475	0.567	0.194
1	0.7	4	1.900	0.555	0.312	0.372	0.195
1	0.35	3	1.450	0.318	0.234	0.282	0.204

or

$$\int_{m_t}^{c_t} (x - m_t).f_{1t}(x)\mathrm{d}x + \int_{c_t}^{b_t} (x - m_t).f_{2t}(x)\mathrm{d}x \quad \text{if } m_t < c_t \tag{20a}$$

$$\int_{m_t}^{b_t} (x - m_t).f_{2t}(x)\mathrm{d}x \quad \text{if } m_t \geq c_t. \tag{20b}$$

Eq. (20a) reduces to

$$\frac{[(b_t - a_t) + (c_t - a_t)]^3}{81(b_t - a_t)(c_t - a_t)} \quad \text{if } m_t < c_t. \tag{21}$$

Eq. (20b) reduces to

$$\frac{[(b_t - a_t) + (b_t - c_t)]^3}{81(b_t - a_t)(b_t - c_t)} \quad \text{if } m_t \geq c_t. \quad \square \tag{22}$$

Table 4 shows in column *Exact* the first bound on the right-hand side of Eq. (15) in case all parameters of the triangular distributions are known, calculated through Eqs. (21) and (22). The column *Bound* shows the last bound on the right-hand side of Eq. (15). This bound is around 20% higher than the bound obtained in the *Exact* column. The sum of the bounds in the *Exact* column equals 7.119 and the sum in the *Bound* column equals 8.587. The column *RelDiff* shows the relative difference calculated as (*Bound* − *Exact*)/*Exact*. The procedure can be applied to other, than the triangular, distributions, as long as the integral in Eq. (19) can be expressed in an analytical way, which is true for the exponential or uniform distributions, but not for the normal and most of the Gamma or Beta distributions. In the latter cases either approximations for the distributions have to be used or bounds computed based on incomplete information (for example based on first or second moments only). This case is elaborated in Section 5. In the next section it is investigated whether even a better bound can be obtained than the one in the *Exact* column.

5. An upper bound on the cycle time in the case of incomplete information

In this section some formulas concerning bounds on the expected value of a random variable are presented when the only knowledge available about the random variable is the bounds of the finite interval and certain integral constraints such as the first and second moments. This section aims to obtain upper bounds for the measure under study through an alternative theoretical basis.

The practical elaboration goes through the use of the following corollary, proven by Brockett and Cox Jr. [2].

Corollary 1. *If the mean μ and variance σ^2 are given, then for any function h with $h^{(3)}(x) \geq 0$ and any random variable X on $[a, b]$ with mean μ and variance σ^2, the following bound is tight*

$$h(a)p + h(\xi_1)(1-p) \leq E[h(x)] \leq h(\xi_2)q + h(b)(1-q) \tag{23}$$

where

$$p = \frac{\sigma^2}{\sigma^2 + (a-\mu)^2}, \qquad \xi_1 = \mu + \frac{\sigma^2}{\mu - a}, \qquad \xi_2 = \mu - \frac{\sigma^2}{b-\mu}, \quad \text{and} \quad q = \frac{(b-\mu)^2}{\sigma^2 + (b-\mu)^2}.$$

Using this information, the following corollary can be proven:

Corollary 2. *If the mean m_t and variance σ_t^2 are given, then for a function $\mathrm{E}\left[(X_t - z_t)^+\right]$ and any random variable X_t on $[a_t, b_t]$ with mean m_t and variance σ_t^2 and $z_t \leq b_t$, the following bound is tight*

$$\mathrm{E}\left[(X_t - z_t)^+\right] \leq B_t \tag{24}$$

where

$$B_t = \begin{cases} (b_t - z_t)\dfrac{\sigma_t^2}{\sigma_t^2 + (b_t - m_t)^2} & \text{if } z_t \geq m_t - \dfrac{\sigma_t^2}{b_t - m_t} \\ m_t - z_t & \text{if } z_t \leq m_t - \dfrac{\sigma_t^2}{b_t - m_t}. \end{cases} \tag{25}$$

Proof. Using the second part of (23) and putting $h(x) = (X_t - z_t)^+$, we get that

$$\mathrm{E}\left[(X_t - z_t)^+\right] \leq (\xi_2 - z_t)^+ q_t + (b_t - z_t)^+(1 - q_t). \tag{26}$$

If $z_t \leq b_t$, this reduces to

$$\mathrm{E}\left[(X_t - z_t)^+\right] \leq (\xi_2 - z_t)^+ q_t + (b_t - z_t)(1 - q_t). \tag{27}$$

This equation splits into two cases depending on whether z_t is greater or smaller than ξ_2.
If $z_t \geq \xi_2$, then $(\xi_2 - z_t)^+ = 0$. In that case

$$\mathrm{E}\left[(X_t - z_t)^+\right] \leq (b_t - z_t)\left(1 - \frac{(b_t - m_t)^2}{\sigma_t^2 + (b_t - m_t)^2}\right) \tag{28}$$

$$= (b_t - z_t)\frac{\sigma_t^2}{\sigma_t^2 + (b_t - m_t)^2}. \tag{29}$$

If $z_t < \xi_2$, then $(\xi_2 - z_t)^+ = (\xi_2 - z_t)$. Therefore

$$\mathrm{E}\left[(X_t - z_t)^+\right] \leq \left(m_t - \frac{\sigma_t^2}{b_t - m_t} - z_t\right)\frac{(b_t - m_t)^2}{\sigma_t^2 + (b_t - m_t)^2} + (b_t - z_t)\frac{\sigma_t^2}{\sigma_t^2 + (b_t - m_t)^2} \tag{30}$$

$$= \frac{1}{\sigma_t^2 + (b_t - m_t)^2}\left[\left(m_t - \frac{\sigma_t^2}{b_t - m_t}\right)(b_t - m_t)^2\right. \tag{31}$$

$$\left. + \, b_t\sigma_t^2 - z_t\left[(b_t - m_t)^2 + \sigma_t^2\right]\right] \tag{32}$$

$$= \frac{1}{\sigma_t^2 + (b_t - m_t)^2}\left[m_t b_t^2 - 2m_t^2 b_t + \mu_t^3 + m_t\sigma_t^2 - z_t\left[(b_t - m_t)^2 + \sigma_t^2\right]\right] \tag{33}$$

$$= \frac{1}{\sigma_t^2 + (b_t - m_t)^2}\left[m_t\left[(b_t - m_t)^2 + \sigma_t^2\right] - z_t\left[(b_t - m_t)^2 + \sigma_t^2\right]\right] \tag{34}$$

$$= m_t - z_t. \tag{35}$$

If we define B_t according to the definition in Eq. (25), the corollary has been proven. □

From this, it is easy to prove the following.

Corollary 3. *If mean m_t and variance σ_t^2 are given for each transition $t \in T$, then the following bound is tight and therefore cannot be improved.*

$$\pi(M_0) \leq \pi^D(M_0) + \inf_{\mathbf{z}\in E} \sum_t (b_t - z_t) \frac{\sigma_t^2}{\sigma_t^2 + (b_t - m_t)^2} \tag{36}$$

where

$$E = \left\{ \mathbf{z} | z_t \geq m_t, \forall t \in T \text{ and } \sum_{t\in\gamma} z_t \leq \pi^D(M_0).M_0(\gamma), \forall \gamma \in \Gamma \right\}. \tag{37}$$

Proof. According to Eq. (37), $z_t \geq m_t$. Also, $b_t \geq m_t$ and $\sigma_t^2 \geq 0$. From this, it is clear that $z_t \geq m_t - \frac{\sigma_t^2}{b_t - m_t}$, taking into consideration (25) inequality (36) turns into

$$\pi(M_0) \leq \pi^D(M_0) + \inf_{\mathbf{z}\in E} \left\{ \sum_{t\in T} B_t \right\} \text{ s.t. } \mathbf{z} \in E \tag{38}$$

$$\pi(M_0) \leq \pi^D(M_0) + \inf_{\mathbf{z}\in E} \left\{ \sum_{t\in T} (b_t - z_t) \frac{\sigma_t^2}{\sigma_t^2 + (b_t - m_t)^2} \right\} \text{ s.t. } \mathbf{z} \in E. \quad \square \tag{39}$$

Finally it is recognized that the second term of (36) is a linear objective function and the constraints in (37) are all of the linear type, so it might be interesting to solve the linear program rather than use the bounds in the previous section.

It can be seen that the second term of (36) can be written as a constant and a sum of terms with variables z_t. The linear program is formulated for the stochastic marked graph of Fig. 1.

$$\begin{aligned}
\min\ & 15.61233 - 0.257384z_1 - 0.225806z_2 - 0.195313z_3 - 0.167148z_4 - 0.142857z_5 \\
& -0.124246z_6 - 0.116910z_7 - 0.111625z_8 - 0.111782z_9 - 0.116863z_{10} \\
\text{s.t.}\ & z_1 \geq 9.10 \\
& z_2 \geq 8.20 \\
& z_3 \geq 7.30 \\
& z_4 \geq 6.40 \\
& z_5 \geq 5.50 \\
& z_6 \geq 4.60 \\
& z_7 \geq 3.70 \\
& z_8 \geq 2.80 \\
& z_9 \geq 1.90 \\
& z_{10} \geq 1.45 \\
& z_1 \leq 14.00 \\
& z_2 \leq 13.00 \\
& z_3 \leq 12.00 \\
& z_4 \leq 11.00 \\
& z_5 \leq 10.00 \\
& z_6 \leq 9.00 \\
& z_7 \leq 7.50 \\
& z_8 \leq 6.00 \\
& z_9 \leq 4.00 \\
& z_{10} \leq 3.00 \\
& z_1 + z_4 + z_5 \leq 39.85 \\
& z_1 + z_2 + z_6 + z_7 \leq 39.85 \\
& z_2 + z_3 + z_8 + z_9 \leq 39.85 \\
& z_1 + z_2 + z_3 + z_4 + z_6 + z_8 + z_{10} \leq 39.85 \\
& z_2 + z_3 + z_8 + z_9 + z_{10} \leq 39.85.
\end{aligned} \tag{40}$$

The optimization of the model leads to the following values of the decision variables: $z_1 = 9.1$, $z_2 = 8.2$, $z_3 = 7.3$, $z_4 = 6.4$, $z_5 = 10.0$, $z_6 = 4.6$, $z_7 = 7.5$, $z_8 = 2.8$, $z_9 = 4.0$ and $z_{10} = 1.45$. This solution leads to an objective function value of 5.1169, which is around 40% less than the bound based on only two moments earlier, and around 30% less than the bound with full information on the triangular distribution. Even if this development is based only on a single example, it looks interesting to take some effort in solving a linear program to obtain superior results.

The bounds, as calculated in this section, are applicable in dynamic systems, which can be modelled as a stochastic marked graph and in which the exact calculation of the bound on the cycle time cannot be expressed due to problems in analytical calculations or simply due to lack of information. A project network (without consideration of resources) can serve as an example. Random times of activity durations are based on an optimistic, a most likely and a pessimistic estimate. Even if the literature assumes a Beta distribution behind this, in fact, only the range and maybe the first and second moments are known. The same might be true for scheduling models with random times. Stochastic marked graphs have been used to model ratio-driven manufacturing systems [13] and in cyclic flow lines where identical sets of jobs are repeatedly produced in the same loading and processing sequence [8]. Links have been made between functional models like the IDEF families and Petri nets to model more complex manufacturing systems [6,7]. As long as these models fit into the stochastic marked graph framework the above bounds may help in evaluating uncertainties in performance.

6. Conclusion

Marked graphs are a special type of Petri nets used in modelling dynamic systems in which no choice is allowed from one system state to another. With respect to performance modelling of such systems, times of state transitions can be described by random variables. If the marked graph is cyclic, the expected value of the cycle time is the main measure of interest. Due to computational difficulties only upper bounds on this measure of interest can be obtained. In this study the upper bound, formulated as a linear program with limited information on the probability distribution of the transition times, is compared with an upper bound based on classical inequalities in probability theory. In the example, which is used throughout the paper, the approach with the linear program is shown to be much superior. It can be concluded that it is worthwhile to take some effort in solving the linear program in order to obtain better bounds for the performance measure of interest.

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