ANZIAM J. 44(2002), 121–128

A BÄCKLUND TRANSFORMATION AND NONLINEAR SUPERPOSITION FORMULA FOR THE LOTKA-VOLTERRA HIERARCHY

XING-BIAO HU[1] [2] and JOHAN SPRINGAEL[3]

(Received 24 May, 2000)

Abstract

A hierarchy of bilinear Lotka-Volterra equations with a unified structure is proposed. The bilinear Bäcklund transformation for this hierarchy and the corresponding canonical Lax pair are obtained. Furthermore, the nonlinear superposition formula is proved rigorously.

1. Introduction

Recursion operators and Hirota bilinear forms have played an important role in the development of soliton theory. Recursion operators were first introduced by Olver in 1977 [23] and developed by Fuchssteiner [5] and by Fokas and Santini [4, 24], while Hirota bilinear forms were introduced by Hirota in 1971 [6]. By using recursion operators, we can easily generate a hierarchy of integrable equations. However, recursion operators cannot be applied directly to bilinear equations. Instead recursion operators are characterized by bilinear equations with a unified structure (or canonical form). For example, starting from the isospectral problem

$$\Psi_x = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix} \Psi, \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

[1]Academy of Mathematics and Systems Sciences, State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific Engineering Computing, Academia Sinica, P.O. Box 2719, Beijing 100080, P.R. China; e-mail: hxb@lsec.cc.ac.cn.
[2]Institute of Mathematics and Statistics, University of Kent at Canterbury, Canterbury CT2 7NF, United Kingdom.
[3]Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium; e-mail: jspringa@mach.vub.ac.be.

we can obtain the AKNS hierarchy [1, 2, 22]

$$\begin{pmatrix} q_t \\ r_t \end{pmatrix} = R^n \begin{pmatrix} q_x \\ r_x \end{pmatrix}, \quad n \geq 1,$$

where R is a recursion operator which is given by

$$R = \frac{1}{2i} \begin{pmatrix} -\partial_x + 2q\partial_x^{-1} r & 2q\partial_x^{-1} q \\ -2r\partial_x^{-1} r & \partial_x - 2r\partial_x^{-1} q \end{pmatrix}.$$

By introducing an infinite number of variables $x = t_1, t_2, t_3, \ldots$ and considering q and r to be functions of $t = (t_1, t_2, t_3, \ldots)$ we have the equivalent equation

$$\begin{pmatrix} q_{t_{n+1}} \\ r_{t_{n+1}} \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} -\partial_x + 2q\partial_x^{-1} r & 2q\partial_x^{-1} q \\ -2r\partial_x^{-1} r & \partial_x - 2r\partial_x^{-1} q \end{pmatrix} \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix}, \quad n \geq 1.$$

Using the dependent variable transformation $q = \sigma/\tau$, $r = \rho/\tau$ one can deduce the bilinear equations, which have a unified structure [12, 22]:

$$\left(D_{t_{n+1}} - \frac{i}{2} D_{t_1} D_{t_n}\right) \sigma \cdot \tau = 0, \quad \left(D_{t_{n+1}} + \frac{i}{2} D_{t_1} D_{t_n}\right) \rho \cdot \tau = 0, \quad D_{t_1}^2 \tau \cdot \tau = -2\sigma\rho,$$

where the Hirota bilinear operators $D_x^m D_t^n$ are defined as [8, 10, 19]

$$D_x^m D_t^n a(x,t) \cdot b(x,t) \equiv (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n a(x,t) b(x',t')|_{x'=x, t'=t}.$$

It should be noted that the unified bilinear form (UBF) for the AKNS hierarchy was obtained by Newell [22] without explicit use of the recursion operator.

There are two systematic ways to obtain such a UBF; one way is the so called recursion operator approach [11, 12] and the other is based on the structure of the soliton solutions [25]. As a result UBFs for several hierarchies of integrable equations can be obtained. Since UBFs are candidates for recursion operators in bilinear form, it is natural to derive such unified bilinear forms by using recursion operators where they are available. Compared with the second method, this also avoids tedious calculations in testing multi-soliton solutions.

On the other hand, the remarkable advantage in finding UBFs based on the structure of the soliton solutions is that this approach does not depend on knowledge of the recursion operator. This can lead in some cases to an unknown recursion operator.

In this paper, we will generalize UBFs to the case of differential-difference equations. By using a corresponding recursion operator, a UBF for the Lotka-Volterra hierarchy is proposed. To our knowledge, this is the first example in literature giving a UBF in the differential-difference case. Furthermore, a bilinear Bäcklund transformation (BBT) for the Lotka-Volterra hierarchy is presented. From this BBT we obtain the Lax pair for the Lotka-Volterra hierarchy in a concise form. Finally, a nonlinear superposition formula is proved rigorously.

2. A UBF for the Lotka-Volterra hierarchy

The Lotka-Volterra (LV) or Kac-van Moerbeke equation is given by

$$u_{n,t_1} = u_n(u_{n-1} - u_{n+1}) \tag{1}$$

with $u_{n,t_1} \equiv \partial_{t_1} u_n$. Much work has been performed on (1) and its generalization (see for example [3, 7, 9, 13–18, 20, 21, 26, 27]). In [27] a recursion operator for (1) was presented in the following form:

$$R = u_n(1 + T_-)(u_n T_- - T_+ u_n)(1 - T_-)^{-1} u_n^{-1}$$

where $T_\pm u_n = u_{n\pm 1}$. As a result, higher order LV equations can be written as

$$u_{n,t_k} = R^{k-1} u_{n,t_1}, \quad k > 1$$

or equivalently

$$u_{n,t_k} = R u_{n,t_{k-1}}. \tag{2}$$

This equation can be bilinearized using the dependent variable transformation

$$u_n = \frac{f_{n-3/2} f_{n+3/2}}{f_{n-1/2} f_{n+1/2}}.$$

Using this last transformation we have that

$$(1 + T_-)^{-1}(u_n^{-1} u_{n,t_k}) = \frac{f_{n+3/2,t_k}}{f_{n+3/2}} - 2\frac{f_{n+1/2,t_k}}{f_{n+1/2}} + \frac{f_{n-1/2,t_k}}{f_{n-1/2}},$$

$$(1 - T_-)^{-1}(u_n^{-1} u_{n,t_{k-1}}) = \frac{f_{n+3/2,t_{k-1}}}{f_{n+3/2}} - \frac{f_{n-1/2,t_{k-1}}}{f_{n-1/2}}.$$

Furthermore, (1) and (2) can respectively be transformed into the bilinear equations

$$\left[D_{t_1} \sinh(\tfrac{1}{2} D_n) + \cosh(\tfrac{3}{2} D_n) - \cosh(\tfrac{1}{2} D_n)\right] f_n \cdot f_n = 0 \tag{3}$$

and

$$\begin{aligned} D_{t_k} f_{n+1/2} \cdot f_{n-1/2} = {} & f_{n+3/2} f_{n-3/2,t_{k-1}} - f_{n+1/2,t_{k-1}} f_{n-1/2} \\ & - f_{n-1/2,t_1} f_{n+1/2,t_{k-1}} + f_{n+1/2,t_1,t_{k-1}} f_{n-1/2}. \end{aligned} \tag{4}$$

Using (3), we can rewrite (4) as

$$\begin{aligned} &\big[D_{t_k} \sinh(\tfrac{1}{2} D_n) + \tfrac{1}{2} D_{t_{k-1}} \sinh(\tfrac{3}{2} D_n) \\ &\quad + \tfrac{1}{2} D_{t_{k-1}} \sinh(\tfrac{1}{2} D_n) - \tfrac{1}{2} D_{t_1} D_{t_{k-1}} \cosh(\tfrac{1}{2} D_n)\big] f_n \cdot f_n = 0, \end{aligned} \tag{5}$$

which is nothing but the UBF of the Lotka-Volterra hierarchy. Equations (3) and (5) together constitute the whole hierarchy of LV equations in bilinear form.

3. A unified BBT and a unified Lax pair for the Lotka-Volterra hierarchy

By application of the exchange formalism one can construct, using the necessary exchange formulas (A1)–(A6) (see Appendix for details) a bilinear Bäcklund transformation for (3) and (5),

$$e^{D_n} f_n \bullet g_n = \lambda f_n g_n + \mu e^{-D_n} f_n \bullet g_n, \tag{6}$$

$$(D_{t_1} - \lambda e^{-D_n} - \gamma) f_n \bullet g_n = 0, \tag{7}$$

$$\left[D_{t_k} + \frac{\lambda}{2} D_{t_{k-1}} e^{-D_n} - \frac{\lambda}{2\mu} D_{t_{k-1}} e^{D_n} - \frac{\lambda^2}{2\mu} D_{t_{k-1}}\right] f_n \bullet g_n = 0, \tag{8}$$

where λ, μ and γ are arbitrary parameters.

Using the linearizing transformation $\psi_n = f_n/g_n$, $v_n = g_{n+1}g_{n-1}/g_n^2$ one can transform the unified BBT (6)–(8) into the following Lax pair:

$$L_n\psi_n \equiv v_n\psi_{n+1} - \lambda\psi_n - \mu v_n \psi_{n-1} = 0, \quad B_n^{(1)}\psi_n \equiv \psi_{n,t_1} - \lambda v_n \psi_{n-1} - \gamma\psi_n = 0,$$

$$B_n^{(k)}\psi_n \equiv \psi_{n,t_k} - \frac{\lambda^2}{\mu} U_{n,t_{k-1}}\psi_n - \lambda v_n (U_{n,t_{k-1}} + U_{n+1,t_{k-1}})\psi_{n-1} - \frac{\lambda^2}{\mu}\psi_{n,t_{k-1}} = 0$$

with $k > 1$ and $U_n = \sum_{l=0}^{\infty} \ln(v_{n-1-l})$. The compatibility condition

$$(L_n B_n^{(k)} - B_n^{(k)} L_n)\psi_n = 0$$

is satisfied when $u_n = v_{n+1/2}v_{n-1/2}$ satisfies (2).

4. A nonlinear superposition formula for the Lotka-Volterra hierarchy

In this section we present a superposition formula for the solutions of the UBF (3) and (5).

PROPOSITION 1. *Let f_0 be a solution of the system given by* (3) *and* (5). *Suppose that f_i ($i = 1, 2$) are two other solutions of* (3) *and* (5) *which are related to f_0 under the unified BBT* (6)–(8) *with parameters $(\lambda_i, \mu_i, \gamma_i)$, that is, $f_0 \overset{(\lambda_i,\mu_i,\gamma_i)}{\longrightarrow} f_i$ ($i = 1, 2$), where $\lambda_1\lambda_2 \neq 0$, $f_j \neq 0$ ($j = 0, 1, 2$). Then f_{12} defined by*

$$e^{-\frac{1}{2}D_n} f_0 \bullet f_{12} = c\left[\lambda_1 e^{-\frac{1}{2}D_n} - \lambda_2 e^{\frac{1}{2}D_n}\right] f_1 \bullet f_2, \tag{9}$$

where c is a nonzero constant, is a new solution which is related to f_1 and f_2 under the unified BBT (6)–(8) *with parameters $(\lambda_2, \mu_2, \gamma_2)$ and $(\lambda_1, \mu_1, \gamma_1)$ respectively.*

PROOF. Similar to the deduction in [14], we can show that

$$f_0 f_{12} = c\left[\mu_1 e^{D_n} + \mu_2 e^{-D_n}\right] f_1 \cdot f_2, \tag{10}$$

$$-D_{t_1} f_1 \cdot f_2 + (\gamma_2 - \gamma_1) f_1 f_2 - (1/c) e^{-D_n} f_0 \cdot f_{12} = 0, \tag{11}$$

$$\left[e^{D_n} - \lambda_j - \mu_j e^{-D_n}\right] f_i \cdot f_{12} = 0, \qquad i \neq j, \quad i, j = 1, 2,$$

$$\left[D_{t_1} - \lambda_j e^{-D_n} - \gamma_j\right] f_i \cdot f_{12} = 0, \qquad i \neq j, \quad i, j = 1, 2.$$

In order to prove Proposition 1, it suffices to show that

$$\left[D_{t_k} + \frac{\lambda_i}{2} D_{t_{k-1}} e^{-D_n} - \frac{\lambda_i}{2\mu_i} D_{t_{k-1}} e^{D_n} - \frac{\lambda_i^2}{2\mu_i} D_{t_{k-1}}\right] f_j \cdot f_{12} = 0, \quad i \neq j, \ i, j = 1, 2. \tag{12}$$

Since f_1 and f_2 are two solutions of (3) and (5), we have that

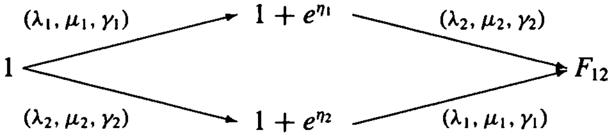
$$\begin{aligned}
&\left[e^{\frac{1}{2}D_n} f_2 \cdot f_2\right] \Big\{\left[D_{t_k} \sinh(\tfrac{1}{2}D_n) + \tfrac{1}{2} D_{t_{k-1}} \sinh(\tfrac{3}{2}D_n) + \tfrac{1}{2} D_{t_{k-1}} \sinh(\tfrac{1}{2}D_n)\right.\\
&\quad \left. - \tfrac{1}{2} D_{t_1} D_{t_{k-1}} \cosh(\tfrac{1}{2}D_n)\right] f_1 \cdot f_1\Big\} - \Big\{\left[D_{t_k} \sinh(\tfrac{1}{2}D_n) + \tfrac{1}{2} D_{t_{k-1}} \sinh(\tfrac{3}{2}D_n)\right.\\
&\quad \left. + \tfrac{1}{2} D_{t_{k-1}} \sinh(\tfrac{1}{2}D_n) - \tfrac{1}{2} D_{t_1} D_{t_{k-1}} \cosh(\tfrac{1}{2}D_n)\right] f_2 \cdot f_2\Big\} \left[e^{\frac{1}{2}D_n} f_1 \cdot f_1\right]\\
&\quad - \tfrac{1}{2}\left[D_{t_{k-1}} e^{\frac{1}{2}D_n} f_2 \cdot f_2\right]\left[D_{t_1} \sinh(\tfrac{1}{2}D_n) + e^{\frac{3}{2}D_n} - e^{\frac{1}{2}D_n}\right] f_1 \cdot f_1\\
&\quad + \tfrac{1}{2}\left[D_{t_{k-1}} e^{\frac{1}{2}D_n} f_1 \cdot f_1\right]\left[D_{t_1} \sinh(\tfrac{1}{2}D_n) + e^{\frac{3}{2}D_n} - e^{\frac{1}{2}D_n}\right] f_2 \cdot f_2 = 0.
\end{aligned} \tag{13}$$

Using formulas (A1)–(A10), (9)–(11) we can rewrite (13) as

$$-\frac{1}{c\lambda_i} e^{-\frac{1}{2}D_n} f_0 f_i \cdot \left\{\left[D_{t_k} + \frac{\lambda_i}{2} D_{t_{k-1}} e^{-D_n} - \frac{\lambda_i}{2\mu_i} D_{t_{k-1}} e^{D_n} - \frac{\lambda_i^2}{2\mu_i} D_{t_{k-1}}\right] f_j \cdot f_{12}\right\} = 0,$$

for $i \neq j$, $i, j = 1, 2$, which means that (12) holds. Therefore we complete the proof of Proposition 1.

As an application of the nonlinear superposition formula (9), we shall now construct soliton solutions of the Lotka-Volterra hierarchy in bilinear form. Choose for example $f_0 = 1$, $c = 1/(\lambda_1 - \lambda_2)$. It is easily verified that



where

$$F_{12} = 1 + \frac{\lambda_1 - \lambda_2 e^{2p_1}}{\lambda_1 - \lambda_2} e^{2\eta_1} + \frac{\lambda_2 - \lambda_1 e^{2p_2}}{\lambda_2 - \lambda_1} e^{2\eta_2} + \frac{\lambda_1 e^{2p_2} - \lambda_2 e^{2p_1}}{\lambda_1 - \lambda_2} e^{2(\eta_1 + \eta_2)}$$

with $\eta_i = -p_i n + \omega_i^{(1)} t_1 + \cdots + \omega_i^{(k)} t_k + \cdots$, $\omega_i^{(1)} = \sinh(2p_i)$, $\lambda_i = 1 + e^{2p_i}$, $\mu_i = -e^{2p_i}$, $\gamma_i = -(1 + e^{2p_i})$ and $\omega_i^{(k)} = (-1)^{k-1} e^{-2(k-1)p_i} (1 + e^{2p_i})^{2(k-1)} \sinh(2p_i)$. In general, along these lines, we can obtain multisoliton solutions for the Lotka-Volterra hierarchy (3) and (5) step by step.

5. Conclusion

In this paper we derived a UBF for the Lotka-Volterra hierarchy through the bilinearization of the recursion operator. Applying the exchange formalism we obtained the corresponding unified BBT leading to a unified Lax pair. Finally, we proved a nonlinear superposition formula for this Lotka-Volterra hierarchy.

Acknowledgements

X. B. Hu would like to thank N. Joshi for the invitation and support to attend the Kruskal 2000 meeting. The results of the present paper were obtained when the first author was visiting the Free University of Brussels. He is grateful to F. Lambert and TENA of the Free University of Brussels for the warm hospitality he received during his short visit. He also greatly acknowledges the support by the Royal Society, the Chinese Academy of Sciences and the National Natural Science Foundation of China (Project no. 10171100). J. S. wishes to acknowledge financial support extended within the framework of the "Interuniversity Poles of Attraction Programme, contract no. P4/08-Belgian State, Prime Minister's Office-Federal Office for Scientific, Technical and Cultural Affairs".

Appendix A. Hirota bilinear operator identities

The following bilinear operator identities hold for arbitrary functions a, b, c and d:

$$\begin{aligned}&[e^{\delta D_n} b\bullet b][D_z \sinh(\delta D_n) a\bullet a] - [e^{\delta D_n} a\bullet a][D_z \sinh(\delta D_n) b\bullet b]\\&\quad = 2\sinh(\delta D_n)(D_z a\bullet b)\bullet ab, \end{aligned}\tag{A1}$$

$$\begin{aligned}&[D_y D_t e^{\frac{1}{2}D_n} a\bullet a][e^{\frac{1}{2}D_n} b\bullet b] - [D_y D_t e^{\frac{1}{2}D_n} b\bullet b][e^{\frac{1}{2}D_n} a\bullet a]\\&\quad - [D_y e^{\frac{1}{2}D_n} a\bullet a][D_t e^{\frac{1}{2}D_n} b\bullet b] + [D_y e^{\frac{1}{2}D_n} b\bullet b][D_t e^{\frac{1}{2}D_n} a\bullet a]\\&\quad = 2D_y \cosh(\tfrac{1}{2}D_n)(D_t a\bullet b)\bullet ab, \end{aligned}\tag{A2}$$

$$\begin{aligned}&[D_y e^{\frac{3}{2}D_n} a\bullet a][e^{\frac{1}{2}D_n} b\bullet b] - [D_y e^{\frac{3}{2}D_n} b\bullet b][e^{\frac{1}{2}D_n} a\bullet a]\\&\quad - [e^{\frac{3}{2}D_n} a\bullet a][D_y e^{\frac{1}{2}D_n} b\bullet b] + [D_y e^{\frac{1}{2}D_n} a\bullet a][e^{\frac{3}{2}D_n} b\bullet b]\\&\quad = 2D_y \cosh(\tfrac{1}{2}D_n)[e^{D_n} a\bullet b]\bullet[e^{-D_n} a\bullet b], \end{aligned}\tag{A3}$$

$$\sinh(\delta D_n) a\bullet a = 0, \tag{A4}$$

$$\begin{aligned}&D_y \cosh(\tfrac{1}{2}D_n)[e^{-D_n} a\bullet b]\bullet ab\\&\quad = -\sinh(\tfrac{1}{2}D_n)\{[D_y e^{-D_n} a\bullet b]\bullet ab + [D_y a\bullet b]\bullet[e^{-D_n} a\bullet b]\}, \end{aligned}\tag{A5}$$

$$\begin{aligned}&D_y \cosh(\tfrac{1}{2}D_n)[e^{D_n} a\bullet b]\bullet ab\\&\quad = \sinh(\tfrac{1}{2}D_n)\{[D_y e^{D_n} a\bullet b]\bullet ab + [D_y a\bullet b]\bullet[e^{D_n} a\bullet b]\}, \end{aligned}\tag{A6}$$

$$2\sinh(\delta D_n)(D_t a\cdot b)\cdot ab = D_t[e^{\delta D_n}a\cdot b]\cdot[e^{-\delta D_n}a\cdot b], \tag{A7}$$

$$D_t[e^{\delta D_n}a\cdot b]\cdot[e^{-\delta D_n}c\cdot d] = e^{\delta D_n}[(D_t a\cdot d)\cdot bc - ad\cdot(D_t c\cdot b)], \tag{A8}$$

$$\begin{aligned} &2D_y\cosh(\tfrac{1}{2}D_n)[e^{-D_n}a\cdot b]\cdot cd \\ &\quad = e^{-\frac{1}{2}D_n}\big\{[D_y e^{-D_n}a\cdot d]\cdot cb - ad\cdot[D_y e^{-D_n}c\cdot b] \\ &\quad + [D_y a\cdot d]\cdot[e^{-D_n}c\cdot b] - [e^{-D_n}a\cdot d]\cdot[D_y c\cdot b]\big\}, \end{aligned} \tag{A9}$$

$$\begin{aligned} &2D_y\cosh(\tfrac{1}{2}D_n)ab\cdot[e^{D_n}c\cdot d] \\ &\quad = e^{-\frac{1}{2}D_n}\big\{[D_y e^{D_n}a\cdot d]\cdot cb - ad\cdot[D_y e^{D_n}c\cdot b] \\ &\quad + [D_y a\cdot d]\cdot[e^{D_n}c\cdot b] - [e^{D_n}a\cdot d]\cdot[D_y c\cdot b]\big\}. \end{aligned} \tag{A10}$$

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